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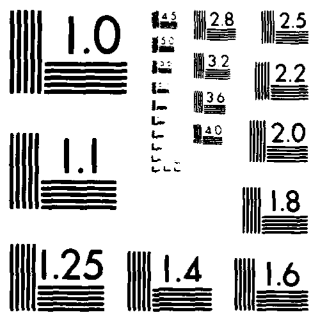

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and
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LEVEL IV

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SINGULARLY PERTURBED EQUATIONS IN THE CRITICAL CASE*

A. B. Vasil'eva and V. F. Butuzov**

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ABSTRACT

This monograph is a sequel to the authors' work "Asymptotic Expansions of Solutions of Singularly Perturbed Equations", Nauka, 1973. It considers cases in which the characterisitc equation has zero roots. In the book many applications are given to concrete physical problems including a detailed examination of several problems in kinetics, in the theory of semiconductors, in numerical difference schemes, etc. The usual mathematical preparation of an engineer is sufficient for understanding the results.

AMS (MOS) Subject Classifications: 34A34, 34B15, 45J05.

Key Words: Singular perturbations, nonlinear initial and boundary value problems, critical case, conditionally stable perturbations, asymptotic expansions, singularly perturbed integro differential equation.

Work Unit Number 1 - Applied Analysis

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**Moscow State University

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Translator's Preface

This is a faithful but not terribly literal translation of the monograph of Professors Vasil'eva and Butuzov which was published by Moscow State University in late 1978. The authors have been gracious enough to send me errata as well as an additional section for Chapter 3. This section is included in the present translation.

I take this opportunity to thank the authors for their most helpful cooperation in this venture, Professor Bob O'Malley for his valuable advice and encouragement, and Professor John Nohel, director of the Research Center, for publishing this translation as an MRC technical report. Finally I would like to thank the National Science Foundation for its generous financial support under grant MCS 78-00907 at the University of Minnesota.

F. A. H.
January 8, 1980
Davis, California

Contents

SINGULARLY PERTURBED EQUATIONS IN THE CRITICAL CASE*

A. B. Vasil'eva and V. F. Butuzov**

Introduction

The present monograph, essentially the sequel to the book "Asymptotic Expansions of Solutions of Singularly Perturbed Equations" [13], is devoted to the asymptotic theory of differential equations with a small parameter μ before the derivative, such as

$$\mu \frac{dz}{dt} = F(z,y,t), \quad \frac{dy}{dt} = f(z,y,t) \quad , \tag{1}$$

and to other related problems involving asymptotic behavior for small μ. Such problems are said to be singularly perturbed.

The difficulty with the construction of the asymptotic expansion of the solution of system (1) arises from the fact that for $\mu = 0$ the order of the system decreases; as a result, the solution of the degenerate system

$$0 = F(\bar{z},\bar{y},t), \quad \frac{d\bar{y}}{dt} = f(\bar{z},\bar{y},t) \quad , \tag{2}$$

cannot in general satisfy all of the supplementary conditions prescribed for (1). On account of this singularity the asymptotic expansion of the solution of system (1) cannot be constructed solely in the form of an "ordinary" series in powers of μ (regular series) but a boundary series must be added whose terms are important only in a neighborhood of those points at which the given supplementary conditions for (1) are not satisfied by (2).

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The techniques for constructing regular and boundary series are described in detail in [13]; in addition, methods for estimating the remainder terms are also discussed there.

All of the problems discussed in [13] were characterized by the fact that the equation $F(\overline{z},\overline{y},t) = 0$ had one or several isolated solutions $\overline{z}$. However, in applications one frequently encounters cases where this equation has a family of solutions which depends on several arbitrary functions. We shall call such cases critical and we shall consider them in the present monograph. Many of these results were obtained in [3,4,9,10,11,12,15]. The critical case can be distinguished analytically by definite signs: some eigenvalues of a special matrix are identically zero. We also discuss several problems of concrete physical importance in a number of fields: problems in kinetics, problems in the theory of semiconductors, numerical difference schemes, etc.

The techniques for constructing the asymptotic expansion in this critical case are basically the same as those in [13]. For a thorough understanding it is advisable for the reader to become acquainted with the first three chapters of that book. However, in the present monograph, for an understanding of at least the formal aspects of the construction of the expansion, it is not necessary to consult [13]. Concerning the estimates for the remainder terms of the asymptotic expansions, the reader should consult [13]. The only exception is Chapter 4 where the proof is somewhat different and so it is given in detail.

The presentation is sufficiently elementary that it can be understood completely by those concerned with applied questions.

Chapter 1

Weakly Nonlinear Singularly Perturbed Equations in the Critical Case: Initial Value Problems

§1 Singularly Perturbed Differential Equations

1. Statement of the Problem. We consider the differential equation

$$\mu \frac{dx}{dt} = A(t)x + \mu f(x,t,\mu) \quad , \tag{1}$$

where $\mu > 0$ is a small parameter, x and f are m-dimensional vector functions, $A(t)$ is an $(m \times m)$-matrix and $0 \leq t \leq T$. A solution of equation (1) should satisfy the initial condition

$$x(0,\mu) = x^0 \quad . \tag{2}$$

If we formally set $\mu = 0$ in (1) then we obtain the reduced equation

$$A(t)\bar{x} = 0 \ . \tag{3}$$

If $\det A(t) \neq 0$ for $0 \leq t \leq T$, equation (3) has the unique solution $\bar{x} \equiv 0$. In [13] it was shown that if the eigenvalues $\lambda_i(t)$ of $A(t)$ satisfy for $0 \leq t \leq T$ the inequalities

$$\operatorname{Re} \lambda_i(t) < 0 \quad (i = 1,\ldots,m) \quad ,$$

then the solution $x(t,\mu)$ of the problem (1), (2) converges as $\mu \to 0$ to $\bar{x} \equiv 0$ for $0 < t \leq T$.

Suppose, however, that $\det A(t) \equiv 0$ for $0 \leq t \leq T$. Then equation (3) has infinitely many solutions and the question arises: under what conditions will the solution $x(t,\mu)$ of the problem (1), (2) converge as $\mu \to 0$ to one of these solutions, and in particular, to which one? The present section is concerned with this question as well as with the question of the construction of the asymptotic expansion of $x(t,\mu)$ with respect to μ.

We impose several additional conditions on equation (1). All of these conditions will not be formulated at the same time but as they are stated in the text. They will be denoted by the numerals I, II, ...

The first condition concerns the smoothness of $A(t)$ and $f(x,t,\mu)$. We require sufficient smoothness in order to construct the desired asymptotic expansion. A more precise formulation of Condition I will be given in Section 3 after we describe the algorithm's construction; until then we formulate this condition as follows.

I. Suppose that $A(t)$ and $f(x,t,\mu)$ are sufficiently smooth for $0 \leq t \leq T$ and for (x,t,μ) in the domain $D(x,t,\mu) = D(x,t) \times [0,\mu_0]$, where $D(x,t)$ is a domain in (x,t)-space and μ_0 is a positive constant.

The following two conditions are concerned with the eigenvalues $\lambda_i(t)$ $(i = 1,\dots,m)$ of $A(t)$. Note that the assumption that $\det A(t) \equiv 0$ for $0 \leq t \leq T$ implies that at least one of the $\lambda_i(t)$ is identically zero.

II. Suppose that for $0 \leq t \leq T$ the following conditions hold:

$$\lambda_i(t) \equiv 0 \quad (i = 1,\ldots,k \ ; \ k < m) \ , \tag{4}$$

$$\operatorname{Re} \lambda_i(t) < 0 \quad (i = k+1,\ldots,m) \ . \tag{5}$$

Remark. In [13] the initial value problem was studied under the assumption that condition (5) was satisfied for all $i = 1,\ldots,m$ (that is, the "noncritical" case). If at least one of the $\lambda_i(t)$ has a positive real part then, generally speaking, the solution of the initial value problem is unbounded as $\mu \to 0$.

III. Suppose that there are k linearly independent eigenvectors $e_i(t)$ $(i = 1,\ldots,k)$ of $A(t)$ corresponding to the k identically zero eigenvalues for each t in $[0,T]$.

Thus we are considering cases where the number of linearly independent eigenvectors corresponding to $\lambda \equiv 0$ is equal to the multiplicity of $\lambda \equiv 0$. For the remaining eigenvalues for which $\operatorname{Re} \lambda_i(t) < 0$, neither their multiplicity nor the number of eigenvectors corresponding to them is of importance; indeed, both of these quantities can change as t varies.

2. Algorithm for the Construction of the Asymptotic Expansion of the Solution.

As we already stated, our goal is the construction of the asymptotic expansion of the solution of problem (1), (2). In order to

achieve this we will follow the same procedure as that adopted previously in the book [13]. First we develop an algorithm (rule) for the construction of certain formal series which determine the structure of the solution. In a neighborhood of the leading terms of these series, there exists a solution of the problem, and the series is itself an asymptotic expansion of this solution. The proof will be given in Section 3 .

Thus we construct a series formally satisfying equation (1) and condition (2), and having the form

$$x(t,\mu) = \bar{x}(t,\mu) + \Pi x(\tau,\mu) \quad , \tag{6}$$

where

$$\bar{x}(t,\mu) = \bar{x}_0(t) + \mu\bar{x}_1(t) + \ldots + \mu^n\bar{x}_n(t) + \ldots \tag{7}$$

is called the regular series, while

$$\Pi x(\tau,\mu) = \Pi_0 x(\tau) + \mu\Pi_1 x(\tau) + \ldots + \mu^n\Pi_n x(\tau) + \ldots \tag{8}$$

is called the boundary series for

$$\tau = t/\mu \quad .$$

The coefficients in the series (7), (8) are determined by formally substituting (6) into (1), (2) and equating terms with like powers of μ according to a definite rule which we state below. First note that the asymptotic expansions in all of the singularly perturbed initial value problems considered in [13] are constructed in the form of series having the structure (6). This structure already occurs in simple examples. Consider, for example, the problem

$$\mu\frac{dx}{dt} = ax + t , \quad x(0,\mu) = x^0 ,$$

where a is a negative constant. The exact solution is $x(t,\mu) = -t/a - \mu/a^2 + (x^0 + \mu/a^2)\exp(a\tau)$, and it consists of terms of the type (7) and (8). Both terms of the latter type converge to zero exponentially as $\tau \to \infty$, and in a neighborhood of $t = 0$ they serve as a correction to the regular part $-t/a - \mu/a^2$ which does not satisfy the given initial condition $x(0,\mu) = x^0$. This structure of (6) reflects such behavior: $\Pi x(\tau,\mu)$ serves as a correction to $\bar{x}(t,\mu)$ in a neighborhood of $t = 0$; moreover, it converges to zero exponentially with increasing τ . The coefficients $\Pi_i x(\tau)$ of the series (8) will be called boundary functions, and we will require that the boundary functions converge to zero as $\tau \to \infty$. Thus the formal algorithm for the construction of the series (7) and (8) requires that

$$\Pi_i x(\tau) \to 0 \quad \text{as} \quad \tau \to \infty . \tag{9}$$

We pass now to the procedure for determining the coefficients in (7), (8). To this end we first represent $f(x,t,\mu)$ in the form

$$f(\bar{x}(t,\mu) + \Pi x(\tau,\mu),t,\mu) = f(\bar{x}(t,\mu),t,\mu)$$
$$+ [f(\bar{x}(\tau\mu,\mu) + \Pi x(\tau,\mu),\tau\mu,\mu) - f(\bar{x}(\tau\mu,\mu),\tau\mu,\mu)] \equiv \bar{f} + \Pi f .$$

Here, by $\bar{f}$ we mean the expansion of $f(\bar{x}(t,\mu),t,\mu)$ in a series of the type (7), while by Πf we mean an expansion of the term in square brackets in a series of the type (8); namely,

$$\bar{f} = \bar{f}_0(t) + \mu\bar{f}_1(t) + \ldots + \mu^n\bar{f}_n(t) + \ldots \quad ,$$

$$\Pi f = \Pi_0 f(\tau) + \mu\Pi_1 f(\tau) + \ldots + \mu^n\Pi_n f(\tau) + \ldots \quad .$$

We perform this same operation on $A(t)x$:

$$A(t)(\bar{x}(t,\mu) + \Pi x(\tau,\mu)) = A(t)\bar{x}(t,\mu) + A(\tau\mu)\Pi x(\tau,\mu) \equiv A\bar{x} + \Pi(Ax) \ .$$

We now substitute (6) into (1) and (2), taking account of the transformations performed on f and Ax :

$$\mu\frac{d}{dt}(\bar{x}_0 + \mu\bar{x}_1 + \ldots) + \frac{d}{d\tau}(\Pi_0 x + \mu\Pi_1 x + \ldots) =$$

$$A\bar{x} + \Pi(Ax) + \mu(\bar{f} + \Pi f) \ , \tag{10}$$

$$\bar{x}(0,\mu) + \Pi x(0,\mu) = x^0 \ . \tag{11}$$

Next we equate coefficients of like powers of μ on both sides of equations (10) and (11), and separating those terms depending on t and those depending on τ, we obtain equations and initial conditions for determining the coefficients $\bar{x}_i(t)$ and $\Pi_i x(\tau)$ of the series (7) and (8) .

For $\bar{x}_0(t)$ we obtain a linear homogeneous system of algebraic equations

$$A(t)\bar{x}_0(t) = 0 \quad , \tag{12}$$

which coincides with the reduced equation (3). By virtue of Condition III the general solution of (12) can be written in the form

$$\bar{x}_0(t) = \sum_{i=1}^{k} \alpha_i(t)e_i(t) \quad , \tag{13}$$

where $e_i(t)$ $(i = 1,\ldots,k)$ are the linearly independent eigenvectors corresponding to the zero eigenvalues of $A(t)$, and $\alpha_i(t)$ are arbitrary scalar functions.

Remark. By virtue of Condition III the rank of the matrix $A(t)$ is equal to $m-k$ for each t in $[0,T]$, that is, there is a minor of order $m-k$ (in general, not the same for all t) which is nonzero, and consequently, the system (12) has k linearly independent solutions (eigenvectors) $e_i(t)$ for $i = 1,\ldots,k$. If this nonzero minor can be found, it is easy to construct eigenvectors $e_i(t)$ having the same degree of smoothness as the matrix $A(t)$. If there is no such minor of $A(t)$ then the question of the degree of smoothness of the $e_i(t)$ becomes more involved. From [27] it follows that it is possible to construct eigenvectors $e_i(t)$ $(i = 1,\ldots,k)$ having the same degree of smoothness as the matrix $A(t)$. Such eigenvectors are used in (13) and below.

If we introduce the $(m \times k)$-matrix $e(t)$ whose columns are these $e_i(t)$ and the k-dimensional vector function $\alpha(t)$ whose components are $\alpha_i(t)$, then it is possible to write (13) in the form

$$\bar{x}_0(t) = e(t)\alpha(t) \quad . \tag{14}$$

For $\Pi_0 x(\tau)$ we obtain a linear homogeneous constant coefficient system of differential equations, namely

$$\frac{d}{d\tau}\Pi_0 x = A(0)\Pi_0 x \quad . \tag{15}$$

The general solution of this system can be written in the form (cf. for example [26])

$$\Pi_0 x(\tau) = \sum_{i=1}^{k} c_i e_i(0) + \sum_{i=k+1}^{m} c_i w_i(\tau)\exp(\lambda_i(0)\tau) \quad , \tag{16}$$

where $c_i (i = 1,\ldots,m)$ are arbitrary constants, $e_i(0)$ $(i = 1,\ldots,k)$ are the eigenvectors of $A(0)$ corresponding to the zero eigenvalues, and $w_i(\tau)$ $(i = k+1,\ldots,m)$ are known vector functions whose components are polynomials in τ . [If $h_1,\ldots,h_n$ is a Jordan chain of vectors corresponding to an eigenvalue λ of A such that

$$Ah_1 = \lambda h_1,\ Ah_2 = \lambda h_2 + h_1, \ldots,\ Ah_n = \lambda h_n + h_{n-1} \ ,$$

then there are n linearly independent solutions $x_r(\tau) = w_r(\tau)\exp(\lambda\tau)$ $(r = 1,\ldots,n)$ of the system $\frac{dx}{d\tau} = Ax$, where

$$w_r(\tau) = \frac{\tau^{r-1}}{(r-1)!} h_1 + \frac{\tau^{r-2}}{(r-2)!} h_2 + \ldots + h_r (r = 1,\ldots,n) \quad .]$$

By virtue of condition (5) the second term in the right-hand side of (16) converges to zero as $\tau \to \infty$. Therefore, in order that condition (9) hold, it is necessary to set $c_i = 0$ $(i = 1,\ldots,k)$.

The initial condition for $\Pi_0 x(\tau)$ is obtained by equating the coefficients of the zeroth power of μ in (11) , namely

$$\Pi_0 x(0) = x^0 - \bar{x}_0(0) = x^0 - \sum_{i=1}^{k} \alpha_i(0) e_i(0) \quad .$$

Substituting now into (16) and noting that $c_i = 0$ $(i = 1,\ldots,k)$ we obtain

$$\sum_{i=1}^{k} \alpha_i(0) e_i(0) + \sum_{i=k+1}^{m} c_i w_i(0) = x^0 \quad . \tag{18}$$

The system (18) is a linear algebraic system of m equations in the m unknowns $\alpha_i(0)$ $(i = 1,\ldots,k)$ and $c_i (i = k+1,\ldots,m)$. By virtue of the linear independence of the column vectors $e_i(0)$ $(i = 1,\ldots,k)$ and $w_i(0)$ $(i = k+1,\ldots,m)$ the system (18) has a unique solution.

Thus $\Pi_0 x(\tau)$ is completely determined. By virtue of (5) it is clear that there exist constants $c > 0$ and $\varkappa > 0$ such that $\|\Pi_0 x(\tau)\| \leq c \exp(-\varkappa\tau)$ for $\tau \geq 0$. [The symbol $\|x\|$ denotes the norm of a vector (matrix) x which is defined, for example, as the sum of the absolute values of its components (elements).] The function $\bar{x}_0(t)$ is not defined until the functions $\alpha_i(t)$ $(i = 1,\ldots,k)$ whose initial values $\alpha_i(0)$ are found from (18) are first defined. Let us set $\alpha(0) = \alpha^0$.

For $\bar{x}_1(t)$ we obtain the linear nonhomogeneous system of algebraic equations

$$A(t)\bar{x}_1(t) = -f(\bar{x}_0(t),t,0) + \frac{d\bar{x}_0(t)}{dt} \equiv \varphi(t) \quad . \tag{19}$$

Since $\det A(t) \equiv 0$ for $0 \leq t \leq T$ a necessary and sufficient condition for the solvability of system (19) is that its right-hand side be orthogonal to each of the eigenvectors $g_j(t)$ $(j = 1,\ldots,k)$ of the adjoint matrix $A^*(t)$ corresponding to the zero eigenvalues. [From linear algebra it is known that the matrix $A^*(t)$, which in the present case is simply

the transpose of $A(t)$, has a zero eigenvalue to which correspond k linearly independent eigenvectors. We note that the degree of smoothness of the vectors $g_j(t)$ is the same as that of the matrix $A(t)$.]

We denote by $\langle a,b \rangle$ the scalar product of two m-dimensional vectors a and b, that is, the sum of products of corresponding components. Thus (taking note of (14)) the solvability condition for (19) can be written as

$$\langle g_j(t), -f(e(t)\alpha(t),t,0) + \frac{d}{dt}(e(t)\alpha(t)) \rangle = 0 , \quad j = 1,\dots,k .$$

We will obtain a system of k nonlinear differential equations for the k unknown functions $\alpha_i(t)$. If we call $g(t)$ the $(k \times m)$-matrix whose rows are the vectors $g_j(t)$ $(j = 1,\dots,k)$, then it is possible to obtain the corresponding system in matrix form, namely

$$(g(t)e(t))\frac{d\alpha}{dt} = g(t)(f(e(t)\alpha(t),t,0) - e'(t)\alpha(t)). \tag{20}$$

The initial values $\alpha(0) = \alpha^0$, as noted above, are found from (18).

From the fact that the number of eigenvectors $e_i(t)$ and $g_j(t)$ is equal to the multiplicity of the zero eigenvalue, it follows that the determinant of the $(k \times k)$-matrix $(g(t)e(t))$ is nonzero. For if this determinant were zero for some t, then a certain nontrivial linear combination of its columns gives a zero column, that is, $\sum_{i=1}^{k} \gamma_i \langle g_j, e_i \rangle = 0$ $(j = 1,\dots,k)$. It follows that $\langle g_j, \sum_{i=1}^{k} \gamma_i e_i \rangle = 0$, that is, the eigenvector $\tilde{e} = \sum_{i=1}^{k} \gamma_i e_i$ of $A(t)$, corresponding to $\lambda = 0$, is

orthogonal to each $g_j (j = 1,\ldots,k)$ so the solvability condition is satisfied for the linear system $Ax = \tilde{e}$. But this implies that the zero eigenvalue of $A(t)$ has adjoint vectors, which contradicts the fact that the number of eigenvectors $e_i(t)$ is equal to the multiplicity of $\lambda = 0$. Thus, the aforementioned determinant is nonzero, and therefore the system (20) can be solved for $\frac{d\alpha}{dt}$, that is,

$$\frac{d\alpha}{dt} = F_0(\alpha,t) \quad , \tag{21}$$

with the form of F_0 being clear from comparing (20) and (21) .

IV. Suppose that the equation (21) with the initial condition $\alpha(0) = \alpha^0$ has a solution $\alpha = \alpha(t)$ for $0 \leq t \leq T$.

Now that $\alpha(t)$ is determined, the solution (14) of the reduced system (12) is complete.

Let us introduce in the space of variables (x,t) a curve L consisting of two components (this curve is the graph of the zeroth pproximation):

$$L_1 = \{(x,t)\colon\ x = \bar{x}_0(0) + \Pi_0 x(\tau)(\tau \geq 0);\ t = 0\} \ ,$$

$$L_2 = \{(x,t)\colon\ x = \bar{x}_0(t);\ 0 \leq t \leq T\} \ .$$

V. Suppose that the curve L lies in the domain $D(x,t)$ appearing in Condition I.

Thus the zeroth order terms in the series (7) and (8) have been determined.

The general solution of the system (19) can be written as

$$\bar{x}_1(t) = \sum_{i=1}^{k} \beta_i(t)e_i(t) + \tilde{x}_1(t) = e(t)\beta(t) + \tilde{x}_1(t) \ , \tag{22}$$

where $\tilde{x}_1(t)$ is a particular solution of (19) and $\beta(t)$ is an arbitrary k-dimensional vector function.

For $\Pi_1x(\tau)$ we obtain a linear nonhomogeneous system of differential equations

$$\begin{aligned} \frac{d\Pi_1 x}{d\tau} = A(0)\Pi_1 x + \tau A'(0)\Pi_0 x(\tau) + f(\bar{x}_0(0) + \Pi_0 x(\tau),0,0) \\ - f(\bar{x}_0(0),0,0) \ . \end{aligned} \tag{23}$$

The initial condition for $\Pi_1x(\tau)$ is found by equating the coefficients of the first power of μ in (11), that is,

$$\Pi_1 x(0) = -\bar{x}_1(0) \ .$$

The general solution of (23) can be written as

$$\Pi_1 x(\tau) = \sum_{i=1}^{k} d_i e_i(0) + \sum_{i=k+1}^{m} d_i w_i(\tau)\exp(\lambda_i(0)\tau) + \tilde{\Pi}_1 x(\tau) \ , \tag{24}$$

where the d_i are arbitrary constants, $w_i(\tau)$ are the same vector functions as those in (16), and $\tilde{\Pi}_1x(\tau)$ is a particular solution of (23) which, it is not difficult to see, can be chosen so that $\|\tilde{\Pi}x_1(\tau)\|$ satisfies the same inequality as $\|\Pi x_0(\tau)\|$, that is, $\|\tilde{\Pi}_1x(\tau)\| \leq c\exp(-\varkappa\tau)$ for $\tau \geq 0$.

<u>Remark</u>. The positive constants c and $\varkappa$ in the estimate for $\tilde{\Pi}_1x(\tau)$ are, in general, not the same as those in the estimate for $\Pi_0x(\tau)$. However, with a view to simplifying the notation, the same notation will be used for analogous constants. Constants of the form c will always

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denote an upper bound on such values, while constants of the form κ will always denote a lower bound.

As in the case of $\Pi_0 x(\tau)$ we require that $d_i = 0$ $(i = 1,\dots,k)$ and then from the initial condition for $\Pi_1 x(\tau)$, we obtain a linear algebraic system of m equations in the m unknowns $\beta_i(0)$ $(i = 1,\dots,k)$ and $d_i (i = k+1,\dots,m)$:

$$\sum_{i=1}^{k} \beta_i e_i(0) + \sum_{i=k+1}^{m} d_i w_i(0) = -\tilde{x}_1(0) - \tilde{\Pi}_1 x(0) \quad , \tag{25}$$

which, like system (18), has a unique solution.

Thus $\Pi_1 x(\tau)$ is completely determined, and it is obvious that $\|\Pi_1 x(\tau)\| \le c \exp(-\kappa\tau)$ for $\tau \ge 0$. The expression for $\bar{x}_1(t)$ is not determined until we determine $\beta(t)$, whose initial value $\beta(0)$ is obtained from (25). It can be determined from a solvability condition for the linear system of algebraic equations relative to $\bar{x}_2(t)$ in a manner analogous to that for $\alpha(t)$. We obtain the linear differential equation

$$\frac{d\beta}{dt} = B(t)\beta + F_1(t) \quad , \tag{26}$$

where $B(t) = \frac{\partial F}{\partial \alpha}(\alpha(t),t)$ is a known matrix and $F_1(t)$ is a known function. By linearity the system (26) with the initial condition $\beta(0)$ has a unique solution.

The determination of the remaining terms in the series (7) and (8) proceeds analogously. At the i-th stage an arbitrary vector function (call it $\gamma(t)$) enters the expression for $\bar{x}_i(t)$. First we determine

$\gamma(0)$ from an equation like (25) and then from the solvability condition for $\bar{x}_{i+1}(t)$ we obtain for $\gamma(t)$ a linear differential equation of the form of (26): $\frac{d\gamma}{dt} = B(t)\gamma + F_i(t)$, from which $\gamma(t)$ is finally determined.

The boundary functions $\Pi_i x(\tau)$ are constructed like $\Pi_1 x(\tau)$ and also satisfy the exponential estimate

$$\|\Pi_i x(\tau)\| \leq c \exp(-\kappa\tau) \quad \text{for} \quad \tau \geq 0 \ . \tag{27}$$

3. <u>An Estimate of the Remainder Term.</u> Let us denote by $X_n(t,\mu)$ the n-th partial sum of the series (6), that is,

$$X_n(t,\mu) = \sum_{i=0}^{n} \mu^i(\bar{x}_i(t) + \Pi_i x(\tau)) \ .$$

We note that for the determination of the terms appearing in $X_n(t,\mu)$ it is sufficient that $A(t)$ and $f(x,t,\mu)$ have continuous partial derivatives with respect to all arguments up to order n inclusive. However, for the proof of Theorem 1.1 below it will be necessary that $A(t)$ and $f(x,t,\mu)$ have a higher degree of smoothness. It is now possible to determine more precisely the extent of the domain $D(x,t)$ appearing in Condition I . Let us introduce a δ-tube of the curve L , that is, the set of points (x,t) whose distance from the curve L does not exceed δ .

For the domain $D(x,t)$ we can take an arbitrary δ-tube of the curve L where δ is independent of μ . Condition I can now be formulated as:

I. Suppose that $A(t)$ in $0 \leq t \leq T$ and $f(x,t,\mu)$ in the domain $D(x,t,\mu) = D(x,t) \times [0,\mu_0]$ have continuous partial derivatives of order $(n + 2)$ inclusive (with respect to each argument).

<u>Theorem 1.1.</u> Under Conditions I - V there exist positive constants μ_0 and c such that for $0 < \mu \leq \mu_0$ the solution $x(t,\mu)$ of the problem (1), (2) exists in the interval $[0,T]$, is unique and satisfies the inequality

$$\|x(t,\mu) - X_n(t,\mu)\| \leq c\mu^{n+1} (0 \leq t \leq T) \quad . \tag{28}$$

(In place of an inequality of the type (28) we will also use the notation $x(t,\mu) - X_n(t,\mu) = \mathcal{O}(\mu^{n+1})$.)

<u>Proof.</u> Let us set $\xi(t,\mu) = x(t,\mu) - X_{n+1}(t,\mu)$. Substituting $x = X_{n+1}(t,\mu) + \xi$ into (1) and (2) we obtain for $\xi(t,\mu)$ the initial value problem

$$\mu \frac{d\xi}{dt} = A(t)\xi + \mu f_x(t,\mu)\xi + G(\xi,t,\mu) \ , \tag{29}$$

$$\xi(0,\mu) = 0 \quad , \tag{30}$$

where $f_x(t,\mu) = f_x(\overline{x}_0(t) + \Pi_0 x(\tau),t,\mu)$ and

$$G(\xi,t,\mu) = A(t)X_{n+1}(t,\mu) + \mu f(X_{n+1}(t,\mu) + \xi,t,\mu)$$
$$- \mu f_x(t,\mu)\xi - \mu \frac{d}{dt} X_{n+1}(t,\mu) \quad .$$

The function $G(\xi,t,\mu)$ has the following two important properties which are established in the same way as in Subsection 4 of Section 10 of [13] :

1. $G(0,t,\mu) = \Theta(\mu^{n+2})$;

2. If $\|\xi_1(t,\mu)\| \le c_1\mu$ and $\|\xi_2(t,\mu)\| \le c_1\mu$ for $0 \le t \le T$ and $0 < \mu \le \mu_1$ (c_1 and μ_1 are certain constants), then there exist constants c_0 and $\mu_0 \le \mu_1$ such that for $0 \le t \le T$ and $0 < \mu \le \mu_0$

$$\|G(\xi_1,t,\mu) - G(\xi_2,t,\mu)\| \le c_0\mu^2 \max_{[0,T]} \|\xi_1 - \xi_2\| \; .$$

When Property 2 holds we will say that $G(\xi,t,\mu)$ is a contraction operator with contraction coefficient of order $\Theta(\mu^2)$ for $\xi = \Theta(\mu)$.

<u>Remark</u>. The constant μ_0 appearing in Property 2 is, generally speaking, not the same as that appearing in the statement of Theorem 1.1. However, for the sake of simplicity of notation we will use the same symbol μ_0 in all bounds involving conditions of sufficient smallness for μ . It is clear a'priori that among all such constants μ_0 the least one will furnish a positive bound.

For the problem (29), (30) we now transform the unknown function by $\xi(t,\mu) = T(t)w(t,\mu)$, where $T(t)$ is a certain differentiable $(m \times m)$-matrix. Thus we obtain the initial value problem

$$\mu\frac{dw}{dt} = T^{-1}(t)A(t)T(t)w + \mu C(t,\mu)w + G_1(w,t,\mu) \quad , \tag{31}$$

$$w(0,\mu) = 0 \quad , \tag{32}$$

where

$$C(t,\mu) = T^{-1}(t)[f_x(t,\mu)T(t) - T'(t)] \quad \text{and}$$

$$G_1(w,t,\mu) = T^{-1}(t)G(T(t)w,t,\mu) \quad .$$

We choose for $T(t)$ a matrix which transforms $A(t)$ in the interval $[0,T]$ into the block-diagonal form, that is,

$$T^{-1}(t)A(t)T(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix} ,$$

where $A_1(t)$ is an $(m-k) \times (m-k)$-matrix whose eigenvalues satisfy $\operatorname{Re}\lambda_i(t) < 0 \; (i = k+1,\ldots,m)$ and $A_2(t)$ is a $(k \times k)$-matrix with k zero eigenvalues $\lambda_i(t) \equiv 0 \quad (i = 1,\ldots,k)$. Such a nonsingular matrix $T(t)$ exists and it is as differentiable as $A(t)$ (cf. [27]). Indeed, in our case, $A_2(t) \equiv 0$ for $0 \leq t \leq T$.

We set

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \; G_1(w,t,\mu) = \begin{pmatrix} G_2(u,v,t,\mu) \\ G_3(u,v,t,\mu) \end{pmatrix}$$

(where u and G_2 have $(m-k)$ - (and v and G_3 k-) components), and we divide the matrix $C(t,\mu)$ into compatible blocks

$$C(t,\mu) = \begin{pmatrix} C_{11}(t,\mu) & C_{12}(t,\mu) \\ C_{21}(t,\mu) & C_{22}(t,\mu) \end{pmatrix} ,$$

where the C_{ij} are continuous and bounded for $0 \leq t \leq T$, $0 < \mu \leq \mu_0$. From (31), (32) we obtain

$$\mu\frac{du}{dt} = A_1(t)u + \mu C_{11}(t,\mu)u + \mu C_{12}(t,\mu)v + G_2(u,v,t,\mu) \ , \tag{33}$$

$$\begin{aligned} \frac{dv}{dt} &= C_{21}(t,\mu)u + C_{22}(t,\mu)v + \frac{1}{\mu}G_3(u,v,t,\mu) \ , \\ u(0,\mu) &= v(0,\mu) = 0 \ . \end{aligned} \tag{34}$$

The functions $G_2(u,v,t,\mu)$ and $G_3(u,v,t,\mu)$ have the same two properties as $G(\xi,t,\mu)$.

Let us denote the fundamental matrices of the homogeneous linear systems

$$\mu\frac{du}{dt} = A_1(t)u \ , \quad \frac{dv}{dt} = C_{22}(t,\mu)v \ ,$$

by $U(t,s,\mu)$ and $V(t,s,\mu)$, respectively ($U(s,s,\mu) = E_{m-k}$, the $(m-k) \times (m-k)$-identity matrix, and $V(s,s,\mu) = E_k$) . Clearly the matrix $V(t,s,\mu)$ is bounded, and since the eigenvalues of $A_1(t)$ have negative real parts, it follows that $U(t,s,\mu)$ satisfies the inequality

$$\|U(t,s,\mu)\| \le c \exp[\frac{-\varkappa(t-s)}{\mu}] \quad \text{for} \quad 0 \le s \le t \le T \ , \ 0 < \mu \le \mu_0 \ .$$

The proof of this estimate is given in Lemma 3.2 of [13] . We now replace the problem (33), (34) with the equivalent system of integral equations

$$\begin{aligned} u(t,\mu) &= \int_0^t U(t,s,\mu)[C_{11}(s,\mu)u(s,\mu) + C_{12}(s,\mu)v(s,\mu) \\ &\qquad + \frac{1}{\mu} G_2(u,v,s,\mu)]ds \ , \\ v(t,\mu) &= \int_0^t V(t,s,\mu)[C_{21}(s,\mu)u(s,\mu) + \frac{1}{\mu} G_3(u,v,s,\mu)]ds \ , \end{aligned} \tag{35}$$

and then by applying the method of successive approximations as in [13; Sec. 10], it is possible to prove that the solution $u(t,\mu), v(t,\mu)$ of (35) exists, is unique, and satisfies the estimate $u(t,\mu) = \mathcal{O}(\mu^{n+1})$, $v(t,\mu) = \mathcal{O}(\mu^{n+1})$.

Thus it follows directly that

$$\xi(t,\mu) = x(t,\mu) - X_{n+1}(t,\mu) = \mathcal{O}(\mu^{n+1}). \tag{36}$$

Since $X_{n+1}(t,\mu) - X_n(t,\mu) = \mu^{n+1}(\bar{x}_{n+1} + \Pi_{n+1}x) = \mathcal{O}(\mu^{n+1})$, from (36) we obtain

$$x(t,\mu) - X_n(t,\mu) = \mathcal{O}(\mu^{n+1})$$

and Theorem 1.1 is proved.

4. Remarks. 1. From (28) and the exponential bound (27) for $\Pi_i x(\tau)$ $(i = 0,1,\ldots,n)$ it follows that $\lim_{\mu\to 0} x(t,\mu) = \bar{x}_0(t)$ for $0 < t \leq T$, that is, the solution $x(t,\mu)$ of the problem (1), (2) converges as $\mu \to 0$ to one of the solutions of the degenerate system. For $0 < t_0 \leq t \leq T$ (t_0 fixed as $\mu \to 0$) $\bar{x}_0(t)$ is the leading term of the asymptotic expansion. In certain problems (see, for example, Subsection 3 of Section 3 below) one is interested in a precise representation of the leading term. Then to determine the initial values $\alpha_i(0)$ of the functions $\alpha_i(t)$ appearing in $\bar{x}_0(t)$ (see (13)) it is convenient to have a system of equations involving the constants c_i of the leading boundary function terms. Such a system can be obtained by taking the

scalar product of (18) with $g_j(0)$ $(j = 1,\ldots,k)$, after which the terms containing c_i vanish, that is ,

$$\sum_{i=1}^{k} \langle g_j(0), e_i(0)\rangle \alpha_i(0) = \langle g_j(0), x^0\rangle \quad (j = 1,\ldots,k) . \tag{37}$$

2. The construction of the asymptotic expansion and the bound on the remainder term have been obtained under the condition that there exist k linearly independent eigenvectors corresponding to the zero eigenvalues of $A(t)$. In the case when the number of linearly independent vectors corresponding to the zero eigenvalue is less than its multiplicity the asymptotic expansion will contain fractional powers of μ . We will not consider the general problem but we will illustrate the occurrence of fractional powers of μ in a model system of two equations,where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} .$$

Suppose that $\det A(t) = a_{11}a_{22} - a_{12}a_{21} \equiv 0$, $a_{11} + a_{22} \equiv 0$ and $a_{11} \neq 0$ for $0 \leq t \leq T$. Thus $\lambda(t) \equiv 0$ is an eigenvalue of $A(t)$ of multiplicity two, to which there corresponds only one eigenvector (since $a_{11} \neq 0$) . In this case (1) has the form

$$\begin{aligned} \mu\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \mu f_1(x_1,x_2,t,\mu) , \\ \mu\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \mu f_2(x_1,x_2,t,\mu) . \end{aligned} \tag{38}$$

Multiplying the first equation by $-a_{21}$ and the second by a_{11} and adding we obtain the following system of equations for $z = x_1$ and $y = -a_{21}x_1 + a_{11}x_2$:

$$\mu \frac{dz}{dt} = \frac{a_{12}}{a_{11}} y + \mu f_1(z, \frac{y+a_{21}z}{a_{11}}, t, \mu) \quad ,$$

$$\frac{dy}{dt} = -a_{21} f_1(z, \frac{y+a_{21}z}{a_{11}}, t, \mu) + a_{11} f_2(z, \frac{y+a_{21}z}{a_{11}}, t, \mu)$$

$$- a_{21}' z - a_{11}' \frac{y+a_{21}z}{a_{11}} \quad ,$$

that is, a system of the form

$$\mu \frac{dz}{dt} = a(t)y + \mu F(z,y,t,\mu) \ , \ \frac{dy}{dt} = G(z,y,t,\mu) \ . \tag{39}$$

The behavior of solutions of system (39) depends critically on the sign of $a(t)$. Let us transform (39) by setting $z = z_1$, $y = \sqrt{\mu}\, z_2$. We obtain the system

$$\begin{aligned} \sqrt{\mu} \frac{dz_1}{dt} &= a(t) z_2 + \sqrt{\mu}\, F(z_1, \sqrt{\mu}\, z_2, t, \mu) , \\ \sqrt{\mu} \frac{dz_2}{dt} &= G(z_1, \sqrt{\mu}\, z_2, t, \mu) \ . \end{aligned} \tag{40}$$

The corresponding characteristic equation (cf. (3.21) in [13]) has the form

$$\begin{vmatrix} -\lambda & a(t) \\ G_{z_1}(\varphi(t),0,t,0) & -\lambda \end{vmatrix} = \lambda^2 - a(t) G_{z_1}(\varphi(t),0,t,0) = 0 \quad ,$$

where $z_1 = \varphi(t)$ is a root of the equation $G(z_1,0,t,0) = 0$.

If $aG_{z_1} < 0$ then λ will be purely imaginary and in the system one usually sees oscillations. Solutions of the initial value problem do not have a limit as $\mu \to 0$, but are bounded and oscillate with a frequency of order $\frac{1}{\sqrt{\mu}}$ (cf. [19]). If however $aG_{z_1} > 0$ the roots λ will have different signs (the so-called conditionally stable case) and consequently solutions will not generally be bounded as $\mu \to 0$. Nonetheless, if we impose appropriate boundary conditions on (38), it is possible to carry out the construction of the asymptotic expansion of the solution as in [13, Sec. 14]. Such an expansion, as can be seen from the way (40) is written, will contain powers of $\sqrt{\mu}$. This case is considered in more detail in [24] .

3. Let us now consider the equation

$$\mu^2 \frac{dx}{dt} = A(t)x + \mu f(x,t,\mu) \quad .$$

It differs from (1) in that the term multiplying the derivative is μ^2 and not μ . Thus the coefficient multiplying the derivative is of a greater degree of smallness than the coefficient multiplying the nonlinear term $f(x,t,\mu)$. As a result, the asymptotic expansion of the solution of the initial value problem contains [along with the regular part $\bar{x}(t,\mu) = \bar{x}_0(t) + \mu\bar{x}_1(t) + \dots$ and the boundary function $\Pi x(\tau,\mu) = \Pi_0 x(\tau) + \mu\Pi_1 x(\tau) + \dots$, depending on $\tau = t/\mu$] the boundary function $Px(s,\mu) = P_0x(s) + \mu P_1x(s) + \dots$, depending on $s = t/\mu^2$.

The conditions on the matrix $A(t)$ are the same as those in Subsection 1. Consequently,

$$\bar{x}_0(t) = e(t)\alpha(t) ,$$

where the function $\alpha(t)$ is defined as in Subsection 2 by a solvability condition in the equation for $\bar{x}_1(t)$; this time however we do not obtain a differential equation for $\alpha(t)$ but rather an algebraic equation

$$g(t)f(e(t)\alpha(t),t,0) = 0 ,$$

where $g(t)$ is the same matrix as in Subsection 2.

$\Pi_0 x(\tau)$ is obtained not from a differential equation but from the algebraic equation $A(0)\Pi_0 x(\tau) = 0$. It follows that

$$\Pi_0 x(\tau) = e(0)h(\tau) ,$$

where $h(\tau)$ satisfies the differential equation

$$\frac{dh}{d\tau} = (g(0)e(0))^{-1}[f(\bar{x}_0(0) + e(0)h(\tau),0,0) - f(\bar{x}_0(0),0,0)]$$

obtained from a solvability condition for $\Pi_1 x(\tau)$.

For $P_0 x(s)$ we obtain the equation

$$\frac{dP_0(s)}{ds} = A(0)P_0 x$$

with the initial condition

$$P_0 x(0) = x^0 - \bar{x}_0(0) - e(0)h(0)$$

($h(0)$ is as yet unknown). Since this equation is like (15) its general solution can be written in the form of (16) :

$$P_0x(s) = \sum_{i=1}^{k} c_i e_i(0) + \sum_{i=k+1}^{m} c_i w_i(0)\exp(\lambda_i(0)s) \quad .$$

From the condition that $P_0x(s) \to 0$ as $s \to \infty$ we have $c_i = 0 (i = 1,\dots,k)$ and so the initial conditions imply

$$e(0)h(0) + \sum_{i=k+1}^{m} c_i w_i(0) = x^0 - \bar{x}_0(0)$$

for the determination of $h(0)$ and c_i $(i = k+1,\dots,m)$. Likewise the function $P_0x(s)$ will be determined completely, and by finding initial conditions for $h(\tau)$ we can finally determine this function $h(\tau)$ from its differential equation.

The essential role in the construction is played by the $(k \times k)$-matrix

$$(g(t)e(t))^{-1}g(t)f_x(\bar{x}_0(t),t,0)e(t) \quad .$$

We require that its eigenvalues $\nu_i(t)$ satisfy

$$\operatorname{Re} \nu_i(t) < 0 \quad (i = 1,\dots,k \; ; \; 0 \le t \le T \;) \quad .$$

If this condition is fulfilled together with certain others it is possible to prove the validity of this asymptotic expansion with the boundary functions having the exponential bounds

$$\|\Pi_i x(\tau)\| \le c \exp(-\kappa\tau) \; (\tau \ge 0) \; ;$$
$$\|P_i x(s)\| \le c \exp(-\kappa s) \; (s \ge 0) \quad .$$

A more detailed consideration of this problem has been given in [5] .

§2 Difference Equations with Small Stepsize

In this section we will consider the difference equation

$$x(t+\mu) = B(t)x(t) + \mu f(x(t),t,\mu) \; , \tag{41}$$

in which x is an m-vector and the argument t varies discretely with small stepsize μ , that is, $t = 0 , \mu, 2\mu, \ldots (t \leq T)$. Such variations of the argument occur, for example, in difference schemes for many integro-differential equations (cf. §3). For simplicity of notation we will write $x(t)$ in place of $x(t,\mu)$. We prescribe the initial condition

$$x(0) = x^0 \; . \tag{42}$$

I. Suppose that the $(m \times m)$-matrix $B(t)$ has for $0 \leq t \leq T$ the eigenvalue $\lambda(t) \equiv 0$ of multiplicity k to which there correspond for each t k linearly independent eigenvectors $e_i(t)$ $(i = 1,\ldots,k)$, while the other eigenvalues satisfy the condition $|\lambda_i(t)| < 1$.

In [13] it was shown that the asymptotic properties of the difference equation (41) in the noncritical case ($k = 0$, that is, all $|\lambda_i(t)| < 1$) are similar to the asymptotic properties of the differential equation (1) in the noncritical case (all $\mathrm{Re}\,\lambda_i(t) < 0$) . We will consider the critical case $(k \neq 0)$; the asymptotic expansion turns out to be similar to the one for the differential equation.

If we define the matrix $A(t) = B(t) - E_m$, then it clearly satisfies Conditions II and III of §1 .

II. Suppose that $B(t)$ and $f(x,t,\mu)$ satisfy the same smoothness conditions as $A(t)$ and $f(x,t,\mu)$ in §1 (cf. Condition I) .

For the problem (41), (42) it is possible to construct an asymptotic expansion in the parameter μ and to give an estimate for the remainder term as in §1. We write the solution in the form (cf. (6) - (8))

$$x(t) = \bar{x}_0(t) + \mu\bar{x}_1(t) + \dots + \Pi_0 x(\tau) + \mu\Pi_1 x(\tau) + \dots \quad . \tag{43}$$

Substituting (43) into (41) and (42) we obtain

$$\begin{aligned} \bar{x}_0(t+\mu) + \mu\bar{x}_1(t+\mu) + \dots + \Pi_0 x(\tau+1) + \mu\Pi_1 x(\tau+1) + \dots = \\ \overline{Bx} + \Pi(Bx) + \mu(\bar{f} + \Pi f) \quad , \end{aligned} \tag{44}$$

$$\bar{x}_0(0) + \mu\bar{x}_1(0) + \dots + \Pi_0 x(0) + \mu\Pi_1 x(0) + \dots = x^0 \; . \tag{45}$$

The right-hand side of (44) reduces to the same form as in (10) . Equating coefficients of like powers of μ in (44) and (45) as in (10) and (11) we obtain for $\bar{x}_0(t)$ a system of equations

$$\bar{x}_0(t) = B(t)\bar{x}_0(t) \quad \text{or} \quad A(t)\bar{x}_0(t) = 0 \quad .$$

It follows that the representation (13) (or (14)) is correct for $\bar{x}_0(t)$, that is,

$$\bar{x}_0(t) = \sum_{i=1}^{k} \alpha_i(t)e_i(t) = e(t)\alpha(t) \quad .$$

<u>Remark</u>. The expression for $\bar{x}_0(t)$ as well as the following equations for $\bar{x}_i(t)$ $(i = 1,2,\dots)$ which appear in the construction of the

asymptotic expansion of the solution will be considered not only for $t = 0, \mu, 2\mu, \ldots$ but also for all t in the interval $[0,T]$. This is necessary as seen already in the equation for $\bar{x}_0(t)$ and in the determination of the subsequent functions $\bar{x}_i(t)$ $(i = 1,2,\ldots)$. The asymptotic representation (43) naturally involves only values of $\bar{x}_i(t)$ for $t = 0, \mu, 2\mu, \ldots$ corresponding to the discrete variation of t in (41).

For $\Pi_0 x(\tau)$ we obtain the system

$$\Pi_0 x(\tau+1) = B(0)\Pi_0 x(\tau) \ , \ \Pi_0 x(0) = x^0 - \bar{x}_0(0) \quad . \tag{46}$$

In contrast to (15) the system (46) is a linear constant coefficient difference system in which the argument τ varies discretely with unit steps. Its solution can be constructed in a manner completely analogous to the construction of the solution of the constant coefficient differential system (cf. [7,21]), namely

$$\Pi_0 x(\tau) = \sum_{i=1}^{k} c_i e_i(0) + \sum_{i=k+1}^{m} c_i w_i(\tau) u_i(\tau) \quad .$$

The $w_i(\tau)$ appearing here are somewhat different than those in the expression (17), but as before, its components are polynomials in τ and the $u_i(\tau)$ are not exponentials as in (16) but solutions of the scalar difference equations

$$u_i(\tau+1) = \lambda_i(0) u_i(\tau), \ u_i(0) = 1 \quad (i = k+1,\ldots,m) \quad .$$

[If $h_1,\dots,h_n$ is a Jordan chain of vectors corresponding to the eigenvalues of the matrix B, then there are n linearly independent solutions $x_r(\tau) = w_r(\tau)\lambda^\tau$, $(r = 1,\dots,n)$, of the system $x(\tau+1) = Bx(\tau)$, where (cf. (17))

$$w_r(\tau) = \frac{\tau(\tau-1)\dots(\tau-r+2)}{\lambda^{r-1}(r-1)!} h_1 + \frac{\tau(\tau-1)\dots(\tau-r+3)}{\lambda^{r-2}(r-2)!} h_2 + \dots + h_r .]$$

Thus,

$$u_i(\tau) = [\lambda_i(0)]^\tau \quad (\tau = 0,1,2,\dots) ,$$

from which it follows by Condition I that $|u_i(\tau)|$ decays exponentially with increasing τ. Taking this into account and proceeding as in §1 we set $c_i = 0$ for $i = 1,\dots,k$ and further, with the aid of equation (18) (as in §1) we determine $\alpha_i(0)$ $(i = 1,\dots,k)$ and $c_i (i = k+1,\dots,m)$.

For $\bar{x}_1(t)$ we obtain the system:

$$\bar{x}_1(t) = B(t)\bar{x}_1(t) + f(\bar{x}_0(t),t,0) - \frac{d\bar{x}_0(t)}{dt} ,$$

coinciding with (19). The solvability condition for this system leads to a system of differential equations (20) for $\alpha(t)$, as described in §1.

III. Suppose that Conditions IV and V of §1 are satisfied.

The construction of further terms is almost exactly as in §1. An unimportant difference occurs only in the fact that in place of (23) we have a nonhomogeneous difference equation

$$\Pi_1 x(\tau+1) = B(0)\Pi_1 x(\tau) + \tau B'(0)\Pi_0 x(\tau)$$

$$+ f(\bar{x}_0(0) + \Pi_0 x(\tau),0,0) - f(\bar{x}_0(0),0,0) \quad ,$$

and in the expression for $\Pi_1 x(\tau)$ we have $u_i(\tau)$ in place of $\exp(\lambda_i(0)\tau)$. Such differences occur in subsequent Π-functions, but for them the exponential estimates

$$\|\Pi_i x(\tau)\| \leq c \exp(-\varkappa\tau) \quad (\tau = 0,1,2,\ldots)$$

hold. Let us denote by $X_n(t,\mu)$ the n-th partial sum of the series (43) .

Theorem 1.2. Under Conditions I-III there exist positive constants μ_0 and c such that for $0 < \mu \leq \mu_0$ a solution $x(t)$ of the problem (41), (42) exists in the interval $[0,T]$, is unique and satisfies the inequality

$$\|x(t) - X_n(t,\mu)\| < c\mu^{n+1} \quad \text{for} \quad t = 0, \mu, 2\mu, \ldots (t \leq T) \quad .$$

Proof. Let us set $\xi(t) = x(t) - X_{n+1}(t,\mu)$; then for $\xi(t)$ we obtain the system

$$\xi(t+\mu) = B(t)\xi(t) + \mu f_x(t,\mu)\xi(t) + G(\xi,t,\mu) \quad ,$$

$$\xi(0) = 0 \quad ,$$

where

$$f_x(t,\mu) = f_x(\bar{x}_0(t) + \Pi_0 x(\tau), t, \mu) \quad \text{and}$$

$$G(\xi,t,\mu) = B(t)X_{n+1}(t,\mu) + \mu f(X_{n+1}(t,\mu) + \xi, t, \mu) - \mu f_x(t,\mu)\xi - X_{n+1}(t,\mu) .$$

Note that $G(\xi,t,\mu)$ has the same two properties as in §1 .

Let us now set

$$w(t) = T^{-1}(t-\mu)\xi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} ,$$

where u has $(m-k)$-components, v k-components, and the matrix $T(t)$ puts $B(t)$ into the block-diagonal form

$$T^{-1}(t)B(t)T(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & E_k \end{pmatrix} .$$

Here E_k is the $(k \times k)$-identity matrix and $B_1(t)$ has eigenvalues $\lambda_i(t)$ satisfying the inequality $|\lambda_i(t)| < 1$.

For u and v we obtain the system of equations

$$u(t+\mu) = B_1(t)u(t) + C_{11}(t,\mu)u(t) + C_{12}(t,\mu)v(t) + G_2(u,v,t,\mu) , \tag{47}$$

$$v(t+\mu) = v(t) + C_{21}(t,\mu)u(t) + C_{22}(t,\mu)v(t) + G_3(u,v,t,\mu) ,$$

$$u(0) = 0, \ v(0) = 0 , \tag{48}$$

where

$$C(t,\mu) = \begin{pmatrix} C_{11}(t,\mu) & C_{12}(t,\mu) \\ C_{21}(t,\mu) & C_{22}(t,\mu) \end{pmatrix}$$

$$= T^{-1}(t)B(t)[T(t-\mu) - T(t)] + \mu T^{-1}(t)f_x(t,\mu) \cdot T(t-\mu)$$

satisfies the estimate

$$C(t,\mu) = \mathcal{O}(\mu) \ , \tag{49}$$

and

$$G_1(u,v,t,\mu) = \begin{pmatrix} G_2(u,v,t,\mu) \\ G_3(u,v,t,\mu) \end{pmatrix} = T^{-1}(t)G(T(t-\mu)\begin{pmatrix} u \\ v \end{pmatrix}, t, \mu)$$

has the same two properties as the function $G(\xi,t,\mu)$.

Let us denote by $U(t,s,\mu)$ and $V(t,s,\mu)$ the corresponding matrix solutions of the homogeneous difference problems

$$U(t+\mu,s,\mu) = B_1(t)U(t,s,\mu) \quad (t = s, s+\mu, \ldots) \ ,$$

$$U(s,s,\mu) = E_{m-k} \ ,$$

and

$$V(t+\mu,s,\mu) = (E_k + C_{22}(t,\mu))V(t,s,\mu)(t = s, s+\mu, \ldots) \ ,$$

$$V(s,s,\mu) = E_k \ .$$

Since

$$\|(E_k + C_{22}(t,\mu))V(t,s,\mu)\| \le (1+c\mu)\|V(t,s,\mu)\|$$

by (49), it follows that

$$\|V(t,s,\mu)\| \le k(1+c\mu)^{(t-s)/\mu} \le k(1+c\mu)^{T/\mu} \le c \quad ,$$

while $U(t,s,\mu)$, by virtue of the fact that the eigenvalues of the matrix $B_1(t)$ have modulus less than one, satisfies the inequality

$$\|U(t,s,\mu)\| \le c\,\exp\left(-\frac{\kappa(t-s)}{\mu}\right) \quad \text{for} \quad t = s, s+\mu, \ldots \tag{50}$$

(cf. [13; Lemma 6.2]) .

Using the matrices $U(t,s,\mu)$ and $V(t,s,\mu)$ we convert the system (47), (48) into the equivalent system of equations $(t = \ell\mu, \ell = 0,1,2,\ldots)$

$$u(t+\mu) = \sum_{i=0}^{\ell} U(t+\mu,(\ell+1-i)\mu,\mu)[C_{11}(t-i\mu,\mu)u(t-i\mu) + C_{12}(t-i\mu,\mu)v(t-i\mu)$$

$$+ G_2(u(t-i\mu),v(t-i\mu),t-i\mu,\mu)] \quad ,$$

$$v(t+\mu) = \sum_{i=0}^{\ell} V(t+\mu,(\ell+1-i)\mu,\mu)[C_{21}(t-i\mu,\mu)u(t-i\mu)$$

$$+ G_3(u(t-i\mu),v(t-i\mu),t-i\mu,\mu)] \quad .$$

[The solution of the difference problem

$$z(t+h) = A(t)z(t) + b(t) \quad (t = 0,h,2h,\ldots), \quad z(0) = z^0 \quad ,$$

can be written in the form $(t = \ell h, \ell = 0,1,2,\ldots)$

$$z(t+h) = \Phi(t+h,0)z^0 + \sum_{i=0}^{\ell} \Phi(t+h,(\ell+1-i)h)b(t-ih) \quad ,$$

where

$$\Phi(t+h,s) = A(t)\Phi(t,s) \quad (t = s,s+h,\ldots) \quad , \quad \Phi(s,s) = E \quad .]$$

By applying the method of successive approximations to this system and using the two properties of G_2 and G_3 as well as the estimates (49), (50), it is not difficult to prove (analogously to [13, §10]) that a solution exists, is unique and satisfies the estimates $u(t) = \Theta(\mu^{n+1})$, $v(t) = \Theta(\mu^{n+1})$. Thus Theorem 1.2 is established.

Remark. Suppose that the stepsize in (41) is equal to μ^2 and not μ, that is, the order of smallness of the stepsize is greater than the coefficient of the nonlinear term $f(x(t),t,\mu)$:

$$x(t+\mu^2) = B(t)x(t) + \mu f(x(t),t,\mu) \quad . \tag{51}$$

Then besides the functions $\bar{x}(t,\mu)$ and $\Pi x(\tau,\mu)$ $(\tau = t/\mu)$ the boundary function $Px(s,\mu)$ $(s = t/\mu^2)$ will occur in the conditions determining the asymptotic solution of the initial value problem, that is, the indicated difference problem is similar in the sense of asymptotic behavior to the differential problem which we discussed in Remark 3 of Subsection 4 in §1. Detailed considerations are given in [5].

§3 Applications

1. Difference Formulas for the Numerical Integration of Differential Equations. Difference formulas for the numerical integration of initial value problems for the scalar differential equation

$$y' = f(y,t), \; y(0) = y^0 \; (0 \leq t \leq T) \tag{52}$$

can be written in the form (cf. [2])

$$\alpha_\ell y_{k+\ell} + \alpha_{\ell-1} y_{k+\ell-1} + \dots + \alpha_0 y_k = h(\beta_\ell f_{k+\ell} + \dots + \beta_0 f_k) \;, \tag{53}$$

where $y = y(ih)$, h is the stepsize, $f_i = f(y_i, ih)$, and α_i, β_i are constants defined by certain equations, one of which is $\sum_{i=0}^{\ell} \alpha_i = 0$. Hence, it follows that the characteristic equation

$$\alpha_\ell \lambda^\ell + \alpha_{\ell-1}\lambda^{\ell-1} + \dots + \alpha_0 = 0$$

corresponding to (53) always has the root $\lambda = 1$. If we write (53) in the form of a system of difference equations of the type (41) (the role of μ is played by h) then the matrix $B(t)$ of the resulting system has the eigenvalue $\lambda = 1$.

Let us consider in more detail one difference scheme of the type (53)

$$y_{k+2} - \frac{3}{2} y_{k+1} + \frac{1}{2} y_k = \frac{1}{4}(5f_{k+1} - 3f_k)$$

(this is called a formula of extrapolation type, of second order ($\ell = 2$) with two steps). Since $\ell = 2$, to use this formula we must prescribe the initial values $y(0)$ and $y(h)$, where $y(h)$ is nearly equal to $y(0)$ by virtue of the smallness of h.

Let us set $y(0) = y^0$, $y(h) = y^0 + hy^1 + h^2 y^2 + \dots$, where $y^1, y^2, \dots$ are certain coefficients. Moreover let us set $t = kh$, $y = y(kh) = x_1(t)$, $y_{k+1} = x_1(t+h) = x_2(t)$. Then we obtain a system

of two difference equations of the form (41) for $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $(t = 0,h,2h,\ldots)$, namely

$$x_1(t+h) = x_2(t) \;,$$
$$x_2(t+h) = -\frac{1}{2}x_1(t) + \frac{3}{2}x_2(t) \tag{54}$$
$$+ \frac{1}{4}h\,[5f(x_2(t),t+h) - 3f(x_1(t),t)]$$

with initial conditions

$$x_1(0) = y^0,\; x_2(0) = y^0 + hy^1 + h^2y^2 + \ldots \;. \tag{55}$$

The matrix $B(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$ of system (54) has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, and so Condition I of §2 holds. A difference between the problems (54), (55) and (41), (42) is that the initial value $x_2(0)$ depends on h; however, this dependence introduces only an insignificant change in the initial conditions for the coefficients of the asymptotic expansion (43), and the algorithm for the construction of the asymptotic solution as well as all estimates remain valid.

The system of equations corresponding to $\overline{x}_0(t) = \begin{pmatrix} \overline{x}_{10}(t) \\ \overline{x}_{20}(t) \end{pmatrix}$ will have the form $\overline{x}_0(t) = B(t)\overline{x}_0(t)$, that is,

$$\overline{x}_{10}(t) = \overline{x}_{20}(t),\; \overline{x}_{20}(t) = -\frac{1}{2}\overline{x}_{10}(t) + \frac{3}{2}\overline{x}_{20}(t) \;;$$

whence, $\overline{x}_0(t) = \alpha(t)e_1 = \alpha(t)\binom{1}{1}$ where $e_1 = \binom{1}{1}$ is the eigenvector

of the matrix $B(t)$ corresponding to $\lambda_1 = 1$ and $\alpha(t)$ is an arbitrary scalar function.

The system of equations for $\Pi_0 x(\tau) = \begin{pmatrix} \Pi_0 x_1(\tau) \\ \Pi_0 x_2(\tau) \end{pmatrix}$ has the form (cf. (46)) $\Pi_0 x(\tau+1) = B(0)\Pi_0 x(\tau)$, that is,

$$\begin{aligned} \Pi_0 x_1(\tau+1) &= \Pi_0 x_2(\tau) \quad , \\ \Pi_0 x_2(\tau+1) &= -\frac{1}{2}\Pi_0 x_1(\tau) + \frac{3}{2}\Pi_0 x_2(\tau) \quad , \end{aligned} \tag{56}$$

with initial conditions

$$\Pi_0 x_1(0) = y^0 - \alpha(0), \; \Pi_0 x_2(0) = y^0 - \alpha(0) \quad . \tag{57}$$

The general solution of (56) can be written as

$$\Pi_0 x(\tau) = c_1 e_1 + c_2 (\tfrac{1}{2})^{\tau} e_2 = c_1 \binom{1}{1} + c_2 (\tfrac{1}{2})^{\tau} \binom{1}{1/2}$$
$$(\tau = 0,1,2,\ldots) \quad ,$$

where $e_2 = \binom{1}{1/2}$ is the eigenvector of $B(t)$ corresponding to $\lambda_2 = 1/2$. We set $c_1 = 0$ in order that $\Pi_0 x(\tau) \to 0$ as $\tau \to \infty$, and substituting $\Pi_0 x(\tau)$ into the initial conditions (57) we obtain

$$c_2 = y^0 - \alpha(0), \; \frac{1}{2}c_2 = y^0 - \alpha(0) \; ;$$

whence, $c_2 = 0$ and $\alpha(0) = y^0$. Thus $\Pi_0 x(\tau) \equiv 0$.

The system of equations corresponding to $\bar{x}_1(t) = \begin{pmatrix} \bar{x}_{11}(t) \\ \bar{x}_{21}(t) \end{pmatrix}$ is

$$\overline{x}_{11}(t) = \overline{x}_{21}(t) - \alpha'(t) ,$$

$$\overline{x}_{21}(t) = -\frac{1}{2}\overline{x}_{11}(t) + \frac{3}{2}\overline{x}_{21}(t) - \alpha'(t) + \frac{1}{2}f(\alpha(t),t) . \qquad (58)$$

Its solvability condition implies the equation

$$\alpha' = f(\alpha,t) . \qquad (59)$$

The initial condition for $\alpha(t)$ was already determined: $\alpha(0) = y^0$. Given that we can find $\alpha(t)$, the construction of the zeroth approximation $X_0(t,h) = \overline{x}_0(t) + \Pi_0 x(\tau) = \overline{x}_0(t)$ is complete. If the construction of the asymptotic solution is continued then boundary functions appear in the following terms, even in the term containing the arbitrary y^1.

We note that the zeroth approximation $X_{10}(t,h)$ for $x_1(t)$ is equal to $\alpha(t)$, as we would expect from considering the exact solution of the initial value problem (52). The solution obtained from the difference scheme therefore generally differs from the exact solution by a term of order h.

If y^1 were not prescribed arbitrarily, but rather so that $y(h) = y^0 + hy^1 + h^2y^2 + \ldots$ differs from the exact solution of (52) by a term of order h^2 (for this it is necessary to set $y^1(0) = f(y^0,0)$) then $\Pi_1 x(\tau)$ and $\overline{x}_{11}(t)$ are equal to zero, while $x_1(t)$, found from the system of difference equations, differs from the exact solution of the problem (52) by a term of order h^2. In fact, by virtue of (59), the system (58) assumes the form

$$\bar{x}_{11}(t) = \bar{x}_{21}(t) - \alpha'(t) \ ,$$

$$\bar{x}_{21}(t) = -\tfrac{1}{2}\bar{x}_{11}(t) + \tfrac{3}{2}\bar{x}_{21}(t) - \tfrac{1}{2}\alpha'(t) \ ,$$

whence

$$\bar{x}_1(t) = \begin{pmatrix} \bar{x}_{11}(t) \\ \bar{x}_{21}(t) \end{pmatrix} = \beta(t) \binom{1}{1} + \binom{0}{\alpha'(t)} \ .$$

The equation for the scalar function $\beta(t)$ is obtained from the solvability condition for the system corresponding to $\bar{x}_2(t) = \begin{pmatrix} \bar{x}_{12}(t) \\ \bar{x}_{22}(t) \end{pmatrix}$, namely

$$\bar{x}_{12}(t) = \bar{x}_{22}(t) - \tfrac{1}{2}\bar{x}''_{10}(t) - \bar{x}'_{11}(t) \ ,$$

$$\bar{x}_{22}(t) = -\tfrac{1}{2}\bar{x}_{12}(t) + \tfrac{3}{2}\bar{x}_{22}(t) - \tfrac{1}{2}\bar{x}''_{20}(t) - \bar{x}'_{21}(t)$$

$$+ \tfrac{1}{4}[5\bar{f}_y(t)\bar{x}_{21}(t) + 5\bar{f}_t(t) - 3\bar{f}_y\bar{x}_{11}(t)] \ .$$

It can be written in the form

$$\beta' = f_y(\alpha(t),t)\beta \ .$$

Since $\Pi_0 x(\tau) \equiv 0$, the equation for $\Pi_1 x(\tau)$ coincides with the equation for $\Pi_0 x(\tau)$, and consequently,

$$\Pi_1 x(\tau) = d_1\binom{1}{1} + d_2(1/2)^{\tau}\binom{1}{1/2} = d_2\left(\tfrac{1}{2}\right)^{\tau}\binom{1}{1/2} \ ,$$

because $d_1 = 0$ in order that $\Pi_1 x(\tau) \to 0$ as $\tau \to \infty$. Substituting for $\Pi_1 x(\tau)$ and $\bar{x}_1(t)$ into the initial conditions

$$\Pi_1 x_1(0) + \bar{x}_{11}(0) = 0 \ , \ \Pi_1 x_2(0) + \bar{x}_{21}(0) = y^1 \ ,$$

we obtain

$$d_2 + \beta(0) = 0 ,$$

$$\frac{1}{2} d_2 + \beta(0) = y^1 - \alpha'(0) = y^1 - f(y^0,0) \quad .$$

Hence if $y^1 = f(y^0,0)$, $d_2 = \beta(0) = 0$, so $\Pi_1 x(\tau) \equiv 0$, $\beta(t) \equiv 0$ and $\bar{x}_{11}(t) \equiv 0$, as stated above.

Remark. For the numerical integration of the singularly perturbed initial value problem

$$\mu \frac{dy}{dt} = f(y,t), \; y(0,\mu) = y^0 (0 \leq t \leq T)$$

we naturally choose a stepsize smaller than μ , for example $h = \mu^2$. Then if we write the difference formula in the form (53) with step $h = \mu^2$, we obtain an equation which reduces to a system of the form (51).

2. Markov Chains. A. A continuous-time, homogeneous Markov process with finitely many states m can be described by the system

$$\frac{dP}{d\tau} = AP, \; P(0) = P^0 , \tag{60}$$

where $P(\tau)$ is a vector with components $p_1(\tau),\ldots,p_m(\tau)$ ($p_i(\tau)$ is the probability of being in the i-th state at time τ), and A is a matrix with constant elements a_{ij} , satisfying $\sum_{i=1}^{m} a_{ij} = 0$. It is known (cf., for example, [20]) that such a matrix A has $\lambda = 0$ as an eigenvalue with as many linearly independent eigenvectors as the multiplicity of the root $\lambda = 0$. We will assume that the other eigenvalues satisfy the condition $\mathrm{Re}\,\lambda_i < 0$ (a so-called proper chain) .

In the study of Markov chains we are interested, in particular, in the limiting behavior of the probabilities $p_i(\tau)$ as $\tau \to \infty$. For the system (60) the limiting values for $\mu \to 0$ and fixed $t = \tau\mu \neq 0$ are found as the components of the solution $P(t,\mu)$ of

$$\mu \frac{dP}{dt} = AP, \quad P(0,\mu) = P^0 \quad .$$

Suppose that $\lambda = 0$ has the eigenvectors $e_1,\ldots,e_k$. Then it follows from §1 that

$$\lim_{\tau \to \infty} P(\tau) = \lim_{\mu \to 0} P(t,\mu) = \overline{P}_0 = \sum_{i=1}^{k} \alpha_i(t) e_i = e\alpha(t) \quad ,$$

where $\alpha(t)$ is determined from equation (20), which for $f = 0$ has the form

$$(ge)\frac{d\alpha}{dt} = 0 \quad ;$$

whence, $\frac{d\alpha}{dt} = 0$, that is, $\alpha(t) \equiv \alpha(0)$. The initial value $\alpha(0)$ can be determined from system (37), which here assumes the (matrix) form

$$(ge)\alpha(0) = P^0 \quad .$$

In particular, if the chain is regular ($\lambda = 0$ is a simple root) then $k = 1$, $e_1 = e = \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}$, while it is possible to take $g_1 = g = (1,\ldots,1)$ by virtue of the condition $\sum_{i=0}^{k} a_{ik} = 0$. Thus

$$\alpha(0) = \sum_{i=1}^{m} p_i^0 \Big/ \sum_{i=1}^{m} e_i = 1\Big/\sum_{i=1}^{m} e_i \quad \text{since} \quad \sum_{i=1}^{m} p_i^0 = 1 \quad ,$$

and

$$\overline{P}_0 = e \Big/ \sum_{i=1}^{m} e_i \quad .$$

Therefore the limiting probability is here independent of the initial state P^0 .

B. A discrete Markov chain can be described by the difference system

$$P(s+1) = BP(s), \; P(0) = P^0 , \tag{61}$$

where $s = 0,1,2,...$ is the number of trials. Now $\sum_{i=1}^{m} b_{ik} = 1$, and so the matrix B has $\lambda = 1$ as an eigenvalue. As before we will assume that this is a proper chain, that is, each eigenvalue different from one satisfies $|\lambda_i| < 1$.

The limiting value as $s \to \infty$ of the probability $P(s)$ coincides with the limiting value as $\mu \to 0$ for fixed $t \neq 0$ of the solution $P(t,\mu)$ of the system $(t = s\mu)$

$$P(t+\mu,\mu) = BP(t,\mu) \quad . \tag{62}$$

It follows from §2 that the limiting value of $P(t,\mu)$ as $\mu \to 0$ can be described as in case A, so for a regular chain

$$\lim_{s\to\infty} P(s) = \lim_{\mu\to 0} P(t,\mu) = \overline{P}_0 = e \Big/ \sum_{i=1}^{m} e_i \quad ,$$

where $e = \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}$ is the eigenvector of B corresponding to $\lambda = 1$.

C. Passage from a discrete process to a continuous one. We consider the Markov chain (61) where we take as independent variables the instants of time t at which the trials occur. We suppose that the trials are separated from each other by a small time interval μ. Introducing t we obtain a system which agrees formally with (62), but for which we naturally assume that the transition probabilities b_{ik} are small (of order μ) for $i \neq k$, but nearly one for $i = k$, that is,

$$b_{ik} = \mu a_{ik}(\mu)(i \neq k),\ b_{ii} = 1 + \mu a_{ii}(\mu),\ a_{ik}(\mu) \to \bar{a}_{ik} \quad ,$$

as $\mu \to 0$. Then (62) assumes the form

$$p_i(t+\mu) = p_i(t) + \mu(a_{i1}p_1(t) + \ldots + a_{im}p_m(t)) \ (i = 1,\ldots,m). \quad (63)$$

This system is of the type (41) where $B = E_m$ has $\lambda = 1$ as an eigenvalue of multiplicity m to which there correspond m eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad e_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} .$$

These will be the vectors $g_j (j = 1,\ldots,m)$.

It follows from §2 that

$$\lim_{\mu \to 0} P(t,\mu) = e\alpha(t) = E_m\alpha(t) = \alpha(t) \quad ,$$

where $\alpha(t)$ is determined from the differential equation (20), which in this case assumes the form

$$\frac{d\alpha}{dt} = \bar{A}\alpha$$

($\bar{A}$ is the matrix with elements $\bar{a}_{ik}$) with initial condition

$$\alpha(0) = g(0)P^0 = P^0 \quad .$$

Thus the limiting value $P(t,\mu)$ satisfies a system of differential equations of the type (60) .

Chapter 2

Nonlinear Singularly Perturbed Equations in the Critical Case: Initial Value Problems

§1 Statement of the Problem and Auxiliary Results

1. Statement of the Problem. In this chapter we consider problems analogous to those of Chapter 1; however, now the nonlinearity in the right-hand side of the equations is not necessarily small:

$$\mu \frac{dx}{dt} = F(x,t,\mu), \quad 0 \leq t \leq T , \tag{1}$$

$$x(0,\mu) = x^0 , \tag{2}$$

where x is an m-dimensional vector. As in Chapter 1 we will consider cases when the reduced equation does not have an isolated solution, but a family of solutions. Consequently the same questions arise as there. Under what conditions will the solution of the problem (1), (2) converge as $\mu \to 0$ to one of the solutions of this family, and in particular, to which one? How can one construct the asymptotic expansion of the solution to an arbitrary order in μ , uniformly valid for t in $[0,T]$? The conditions, under which the asymptotic expansion will be constructed, are numbered I, II, ...

I. Suppose that the function $F(x,t,\mu)$ is sufficiently smooth in the domain $D(x,t,\mu) = D(x,t) \times [0,\mu_0]$, where $D(x,t)$ is a domain in (x,t)-space and μ_0 is a constant.

II. Suppose that the reduced equation

$$0 = F(\bar{x},t,0)$$

has for each t in $[0,T]$ a family of solutions

$$\bar{x} = \varphi(t,\alpha_1,\ldots,\alpha_k) = \varphi(t,\alpha) \quad ,$$

where $\varphi(t,\alpha)$ is a well-defined function of t and the arbitrary parameters $\alpha_1,\ldots,\alpha_k$ which satisfies the following conditions in the domain $D(t,\alpha) = [0,T] \times D(\alpha)$:

1) the function $\varphi(t,\alpha)$ is sufficiently smooth;

2) the rank of the matrix $\varphi_\alpha(t,\alpha) = \frac{\partial\varphi(t,\alpha)}{\partial\alpha}$ is equal to k, the number of parameters.

From Condition II it follows that $F(\varphi(t,\alpha),t,0) \equiv 0$ for (t,α) in $D(t,\alpha)$. Differentiating this identity with respect to α we obtain

$$F_x(\varphi(t,\alpha),t,0)\varphi_\alpha(t,\alpha) \equiv 0 \quad \text{for} \quad (t,\alpha) \quad \text{in} \quad D(t,\alpha) \quad .$$

This implies that the matrix $F_x(\varphi(t,\alpha),t,0)$ has the eigenvalue $\lambda(t,\alpha) \equiv 0$ and that the columns of the matrix $\varphi_\alpha(t,\alpha)$ are eigenvectors corresponding to $\lambda \equiv 0$. By virtue of condition 2) of II these columns are linearly independent since the multiplicity of $\lambda \equiv 0$ is not less than k.

III. Suppose that the multiplicity of the eigenvalue $\lambda \equiv 0$ is exactly equal to k and that the remaining eigenvalues $\lambda_i(t,\alpha)$ of the matrix $F_x(\varphi(t,\alpha),t,0)$ satisfy

$$\operatorname{Re} \lambda_i(t,\alpha) < 0 \tag{3}$$

in $D(t,\alpha)$.

Under Conditions I-III and certain others which will be stated below we will construct the formal asymptotic expansion of the solution. In §3 we estimate the remainder term, while in the next subsection we obtain a number of auxiliary results which follow from analogous results in [13, §14, Subsection 3] .

2. Auxiliary Results. A. Stability Manifolds. An equation which will play an important role in the construction of boundary functions is

$$\frac{dx}{d\tau} = F(\varphi(0,\alpha) + x,0,0) \quad , \tag{4}$$

where α is a parameter. For any α in $D(\alpha)$ this equation has the rest point $x = 0$. By Condition III the characteristic equation corresponding to this rest point, $\det(F_x(\varphi(0,\alpha),0,0) - \lambda E_m) = 0$, has the root $\lambda = 0$ of multiplicity k and $(m-k)$ roots satisfying condition (3) . Therefore, the rest point $x = 0$ is not asymptotically stable in the sense of Lyapunov, that is, a solution with initial value arbitrarily close to the rest point will not necessarily converge to it as $\tau \to \infty$. However, if we prescribe special initial conditions, then the solution will converge exponentially to the rest point $x = 0$ as $\tau \to \infty$. Put precisely, we have

Lemma 1. In a sufficiently small neighborhood of the point $x = 0$ there exists an $(m-k)$-dimensional manifold $\omega(\alpha)$ such that if the

initial values $x(0)$ belong to $\omega(\alpha)$, then one can find positive constants γ and σ such that for $\tau \geq 0$ the solution $x(\tau)$ satisfies the inequality

$$\|x(\tau)\| \leq \gamma \exp(-\sigma\tau) \quad . \tag{5}$$

Proof. We linearize the right-hand side of (4) with respect to x and write the system (4) as

$$\frac{dx}{d\tau} = A(\alpha)x + G(x,\alpha) \quad , \tag{6}$$

where

$$A(\alpha) = F_x(\varphi(0,\alpha),0,0) \quad \text{and} \quad G(x,\alpha) = F(\varphi(0,\alpha) + x,0,0) - A(\alpha)x \quad .$$

The function $G(x,\alpha)$ has the following two important properties:

1. $G(0,\alpha) = F(\varphi(0,\alpha),0,0) = 0$.

2. For any $\epsilon > 0$ there exists a $\delta > 0$ (depending on ϵ and possibly on α) such that if $\|x_1\| \leq \delta$ and $\|x_2\| \leq \delta$ then

$$\|G(x_1,\alpha) - G(x_2,\alpha)\| \leq \epsilon\|x_1 - x_2\| \quad .$$

This inequality is established by elementary means using Taylor's formula, and it shows that for sufficiently small $\|x\|$ $G(x,\alpha)$ is a contraction operator.

As noted above the matrix $A(\alpha)$ has $\lambda = 0$ as an eigenvalue of multiplicity k and $m-k$ eigenvalues $\lambda_i(0,\alpha)$ which satisfy condition (3). Thus there exists a matrix $B(\alpha)$, as smooth as $A(\alpha)$ (cf. [29]) , which reduces $A(\alpha)$ to the block-diagonal form

$$B^{-1}(\alpha)A(\alpha)B(\alpha) = \begin{pmatrix} C(\alpha) & 0 \\ 0 & 0 \end{pmatrix} , \tag{7}$$

where the $((m-k) \times (m-k))$-matrix $C(\alpha)$ has the stable eigenvalues $\lambda_i(0,\alpha)$ satisfying condition (3) .

Let us make the change of variables

$$x = B(\alpha) \binom{u}{v} ,$$

where u has $(m-k)$- and v k-components. For u and v we obtain

$$\frac{du}{d\tau} = C(\alpha)u + G_1(u,v,\alpha), \ \frac{dv}{d\tau} = G_2(u,v,\alpha) , \tag{8}$$

in which G_1 and G_2 are blocks of vectors $B^{-1}(\alpha)G(B(\alpha)\binom{u}{v},\alpha)$ satisfying the same two properties as $G(x,\alpha)$.

We introduce the system of integral equations

$$u(\tau) = U(\tau,\alpha)u^0 + \int_0^\tau U(\tau,\alpha)U^{-1}(s,\alpha)G_1(u(s),v(s),\alpha)\,ds ,$$

and (9)

$$v(\tau) = \int_\infty^\tau G_2(u(s),v(s),\alpha)\,ds ,$$

where $U(\tau,\alpha)$ is a fundamental matrix of the system $\frac{du}{d\tau} = C(\alpha)u\,(U(0,\alpha) = E_{m-k})$ and u^0 is an arbitrary constant vector. The matrix $U(\tau,\alpha)$ satisfies the inequality $\|U(\tau,\alpha)U^{-1}(s,\alpha)\| \leq M \exp(-\kappa(\tau-s))$ (where the positive constants M and κ can depend on α) . Every solution of system (9) also satisfies system (8).

We apply the method of successive approximations to (9), replacing u and v in the right-hand side of (9) by u_n and v_n , and in the left-hand

side by u_{n+1} and v_{n+1}. Taking $u_0 = v_0 = 0$ we obtain $u_1(\tau) = U(\tau,0)u^0$ and $v_1(\tau) = 0$; whence ,

$$\|u_1(\tau)\| \leq M\exp(-\varkappa\tau)\|u^0\| \leq M\|u^0\|\exp(-\sigma\tau) \quad , \tag{10}$$

where σ is any number in the interval $(0,\varkappa)$.

Let us set $\beta = \max(M/(\varkappa-\sigma), 1/\sigma)$ and choose $\epsilon > 0$ so small that $2\beta\epsilon = q < 1$. Corresponding to this ϵ is a certain number δ defined by the second property of the functions G_1 and G_2 . Let us pick $\rho > 0$ so that the inequality $(M/2)(1/(1-q) + 1)\rho < \delta$ holds. Now consider those u^0 with $\|u^0\| \leq \rho$. Using (10) and Property 1 of G_1 and G_2 we obtain

$$\|u_2(\tau) - u_1(\tau)\| \leq \int_0^\tau M\exp(-\varkappa(\tau-s))\epsilon M\|u^0\|\exp(-\sigma s)ds$$

$$\leq (M/2)q\|u^0\|\exp(-\sigma\tau) \quad ,$$

$$\|v_2(\tau) - v_1(\tau)\| \leq \int_\tau^\infty \epsilon M\|u^0\|\exp(-\sigma s)ds$$

$$\leq (M/2)q\|u^0\|\exp(-\sigma\tau) \quad .$$

It is easy to show that for $n \geq 1$

$$\|u_{n+1}(\tau) - u_n(\tau)\| \leq (M/2)q^n\|u^0\|\exp(-\sigma\tau) \quad ,$$

$$\|u_{n+1}(\tau)\| \leq (M/2)(q^n + q^{n-1} + \ldots + 2)\|u^0\|\exp(-\sigma\tau)$$

$$\leq \delta\exp(-\sigma\tau)$$

and likewise for $v_n(\tau)$. The uniform convergence in τ of the successive approximations follows, and this proves the existence of a solution satisfying

$$\|u(\tau)\| \leq \delta \exp(-\sigma\tau) , \|v(\tau)\| \leq \delta \exp(-\sigma\tau) .$$

Hence, (5) follows directly.

The desired manifold $\omega(\alpha)$ has the form

$$\omega(\alpha) = \{x: x = B(\alpha)\binom{u^0}{v^0} , u = u^0 , v^0 = \int_\infty^0 G_2(u(s),v(s),\alpha)\,ds ,$$
$$\|u^0\| \leq \rho\} .$$

This concludes the proof of Lemma 1.

If we consider the linear approximation for system (4), that is, if we set $G(x,\alpha) = 0$ in (6), then system (8) assumes the form

$$\frac{du}{d\tau} = C(\alpha)u , \frac{dv}{d\tau} = 0 ,$$

and consequently, to obtain a solution which converges exponentially to zero as $\tau \to \infty$, it is necessary to take $v(0) = 0$ (whence $v(\tau) \equiv 0$) and $u(0) = u^0$, where u^0 is arbitrary. Let us denote by z and y the upper and lower blocks of the vector x corresponding to the dimensions $m-k$ and k and by $B_{ij}(\alpha)$ the corresponding blocks of the matrix $B(\alpha)$; from the equation $\binom{z}{y} = B(\alpha)\binom{u}{0}$ we obtain $z = B_{11}(\alpha)u$ and $y = B_{21}(\alpha)u$.

IV. Suppose that $\det B_{11}(\alpha) \neq 0$ for α in $D(\alpha)$.

Then for the linear approximation the manifold $\omega(\alpha)$ can be written as

$$y = B_{21}(\alpha) B_{11}^{-1}(\alpha) z \quad . \tag{11}$$

B. Extension of the Stability Manifold.

The statement of Lemma 1 has a local character. If we continue the trajectories originating in $\omega(\alpha)$ in the negative τ-direction we obtain an extended manifold (denoted by $\Omega(\alpha)$) which has the same property as $\omega(\alpha)$, that is, trajectories starting in $\Omega(\alpha)$ at $\tau = 0$ remain in $\Omega(\alpha)$ for $\tau > 0$ and converge exponentially to the rest point $x = 0$ as $\tau \to \infty$. In some cases we can construct $\Omega(\alpha)$ in an explicit form (cf.§§4 and 5 below). We will assume in the present chapter that $\Omega(\alpha)$ admits of the following analytical representation.

V. Suppose that in some domain $D(z,\alpha)$ the manifold $\Omega(\alpha)$ can be represented as

$$y = P(z,\alpha) \quad , \tag{12}$$

where $P(z,\alpha)$ is a sufficiently smooth function.

Indeed, the definition (12) is an identity along trajectories $x(\tau)$ which converge exponentially to the rest point $x = 0$ as $\tau \to \infty$, that is, the manifold $\Omega(\alpha)$ consists of such trajectories. Therefore, along any such trajectory $\frac{dy}{d\tau} = \frac{\partial P(z,\alpha)}{\partial z}\frac{dz}{d\tau}$, and setting $\frac{\partial P(z,\alpha)}{\partial z} = H(z,\alpha)$,

$$\frac{dy}{d\tau} = H(z,\alpha)\frac{dz}{d\tau} \quad . \tag{13}$$

Let us denote $F(\varphi(0,\alpha) + x,0,0)$ by $F(z,y,\alpha,0,0)$ and the upper and lower blocks of F by F_1 and F_2. Then (4) and (13) imply that along the indicated trajectory

$$\frac{dz}{d\tau} = F_1(z,P(z,\alpha),\alpha,0,0) \quad ,$$

$$\frac{dy}{d\tau} = F_2(z,P(z,\alpha),\alpha,0,0) = H(z,\alpha)F_1(z,P(z,\alpha),\alpha,0,0) \quad . \tag{14}$$

Equation (14) is an identity for (z,α) in $D(z,\alpha)$. Differentiating with respect to z we obtain

$$F_{21} + F_{22}H = (\frac{\partial H}{\partial z} F_1) + H(F_{11} + F_{12}H) \quad , \tag{15}$$

where

$$F_{21} = \frac{\partial F_2}{\partial z} \ , \ F_{22} = \frac{\partial F_2}{\partial y} \ , \ F_{11} = \frac{\partial F_1}{\partial z} \ , \ F_{12} = \frac{\partial F_1}{\partial y} \quad ,$$

and $(\frac{\partial H}{\partial z} F_1)$ denotes the $(k \times (m-k))$-matrix with elements $\sum_{\ell=1}^{m-k} \frac{\partial H^{i\ell}}{\partial z^j} F_1^{\ell}$ (the upper indices correspond to the columns of the matrix). From the definition of $H(z,\alpha)$ it follows that

$$\frac{\partial H^{i\ell}}{\partial z^j} = \frac{\partial^2 P^i}{\partial z^j \partial z^\ell} = \frac{\partial^2 P^i}{\partial z^\ell \partial z^j} = \frac{\partial H^{ij}}{\partial z^\ell} \quad .$$

Therefore

$$\sum_{\ell=1}^{m-k} \frac{\partial H^{i\ell}}{\partial z^j} F^{\ell} = \sum_{\ell=1}^{m-k} \frac{\partial H^{ij}}{\partial z^\ell} F^{\ell} \quad .$$

It is clear that the last sum taken along a trajectory $x(\tau)$ in $\Omega(\alpha)$ is equal to $dH^{ij}/d\tau$. Likewise, along the indicated trajectory the equation $(\frac{\partial H}{\partial z} F_1) = dH/d\tau$ holds, and consequently, from (15) we have

$$\frac{dH}{d\tau} = (F_{21} + F_{22}H) - H(F_{11} + F_{12}H) \tag{16}$$

along any trajectory $x(\tau)$ in $\Omega(\alpha)$.

C. The Variational System on the Stability Manifold. We consider now a nonhomogeneous system of equations whose homogeneous part is the variational system for (4), namely

$$\frac{d\Delta}{d\tau} = F_x(\tau)\Delta + \psi(\tau) \quad , \tag{17}$$

where $F_x(\tau) = F_x(\varpi(0,\alpha) + x(\tau),0,0)$, $x(\tau) \in \Omega(\alpha)$ for some constant vector α in $D(\alpha)$, and $\psi(\tau)$ is a certain function. The upper and lower blocks of Δ having dimensions $m-k$ and k are denoted by Δ_1 and Δ_2, while the corresponding blocks of $\psi(\tau)$ are denoted by $\psi_1(\tau)$ and $\psi_2(\tau)$.

<u>Lemma 2</u>. The change of variables

$$\Delta_1 = \delta_1, \; \Delta_2 = H(\tau)\delta_1 + \delta_2 \; , \tag{18}$$

where $H(\tau) = H(z(\tau),\alpha)$ and $z(\tau)$ is the upper block of $x(\tau)$, transforms the system (17) into the form

$$\frac{d\delta_1}{d\tau} = a_{11}(\tau)\delta_1 + a_{12}(\tau)\delta_2 + \psi_1(\tau) \quad , \tag{19}$$

$$\frac{d\delta_2}{d\tau} = a_{22}(\tau)\delta_2 + (\psi_2(\tau) - H(\tau)\psi_1(\tau)) \quad ,$$

where

$$a_{11}(\tau) = F_{11}(\tau) + F_{12}(\tau)H(\tau), \; a_{12}(\tau) = F_{12}(\tau) \; ,$$

$$a_{22}(\tau) = F_{22}(\tau) - H(\tau)F_{12}(\tau) \; ; \tag{20}$$

here the F_{ij} are the blocks of the matrix $F_x(\tau)$, as in (15) .

The essence of the lemma is that by this change of variables the equation for δ_2 can be separated from the equation for δ_1 . To prove Lemma 2 it is necessary to write system (17) in block form, make the change of variables, and use the identity (16).

Suppose now that the nonhomogeneous term $\psi(\tau)$ in system (17) is $F_x(\tau)\varphi_\alpha(0,\alpha)$. Then the second equation of (19) assumes the form

$$\frac{d\delta_2}{d\tau} = a_{22}(\tau)\delta_2 + [F_x(\tau)\varphi_\alpha(0,\alpha)]_2 - H(\tau)[F_x(\tau)\varphi_\alpha(0,\alpha)]_1 \tag{21}$$

(here and below the indices 1 and 2 denote the upper m-k and the lower k rows of the indicated matrix).

A particular solution of system (17) for the given nonhomogeneous term is clearly $\Delta = -\varphi_\alpha(0,\alpha)$. The corresponding particular solution of (21) is

$$\delta_2 = H(\tau)[\varphi_\alpha(0,\alpha)]_1 - [\varphi_\alpha(0,\alpha)]_2 \equiv R(\tau,\alpha) \quad .$$

Thus we have proved

Lemma 3. The matrix $\delta_2 = R(\tau,\alpha)$ is a solution of equation (21) satisfying the initial condition

$$\delta_2(0) = H(0)[\varphi_\alpha(0,\alpha)]_1 - [\varphi_\alpha(0,\alpha)]_2 = R(0,\alpha) \quad .$$

D. We now obtain a number of important results for the matrices $a_{11}(\tau)$ and $a_{22}(\tau)$ (cf. (20)). Let us denote by $H(\infty)$ the limiting value of $H(\tau) = H(z(\tau),\alpha)$ as $\tau \to \infty$. Since (11) is the linear approximation for (12) we have that

$$\frac{\partial P}{\partial z}(0,\alpha) = H(\infty) = B_{21}(\alpha)B_{11}^{-1}(\alpha) \quad . \tag{22}$$

Let us denote by $F_{ij}(\infty)$ the limiting value of $F_{ij}(\tau)$. Then $F_{ij}(\infty)$ are the blocks of the matrix $F(\infty) = F_x(\varphi(0,\alpha),0,0) = A(\alpha)$ appearing in equation (7). We write (7) as

$$F_x(\infty)B(\alpha) = B(\alpha)\begin{pmatrix} C(\alpha) & 0 \\ 0 & 0 \end{pmatrix}$$

and by equating blocks with index 11 we obtain

$$F_{11}(\infty)B_{11}(\alpha) + F_{12}(\infty)B_{21}(\alpha) = B_{11}(\alpha)C(\alpha) \quad .$$

On account of (20) and (22), it follows that

$$F_{11}(\infty) + F_{12}(\infty)H(\infty) = a_{11}(\infty) = B_{11}(\alpha)C(\alpha)B_{11}^{-1}(\alpha) \quad .$$

Clearly the eigenvalues of $a_{11}(\infty)$ coincide with those of $C(\alpha)$; they are the $\lambda_i(0,\alpha)$ which satisfy condition (3). Thus, the fundamental matrix $\Phi(\tau)$ of $d\delta_1/d\tau = a_{11}(\tau)\delta_1$ ($\Phi(0) = E_{m-k}$) satisfies (cf. [13, (3.78)])

$$\|\Phi(\tau)\Phi^{-1}(s)\| \leq c\exp(-\kappa(\tau-s)) \quad \text{for} \quad 0 \leq s \leq \tau . \tag{23}$$

Analogously we can write (7) as

$$B^{-1}(\alpha)F_x(\infty) = \begin{pmatrix} C(\alpha) & 0 \\ 0 & 0 \end{pmatrix} B^{-1}(\alpha) ,$$

and by equating the blocks with index 22 we obtain

$$(B^{-1}(\alpha))_{21}F_{12}(\infty) + (B^{-1}(\alpha))_{22}F_{22}(\infty) = 0 .$$

Since

$$(B^{-1}(\alpha))_{21} = -(B^{-1}(\alpha))_{22}B_{21}(\alpha)B_{11}^{-1}(\alpha) \tag{24}$$

and

$$\det(B^{-1}(\alpha))_{22} \neq 0 \tag{25}$$

(cf. [20] or [13, (4.55)]),

$$\begin{aligned} F_{22}(\infty) - B_{21}(\alpha)B_{11}^{-1}(\alpha)F_{12}(\infty) &= F_{22}(\infty) - H(\infty)F_{12}(\infty) \\ &= a_{22}(\infty) = 0 . \end{aligned}$$

Now $F_{ij}(\tau) = F_{ij}(\varphi(0,\alpha) + x(\tau),0,0)$ converges exponentially to $F_{ij}(\infty)$ as $\tau \to \infty$ by virtue of the exponential convergence of $x(\tau)$ to zero, and so it follows that

$$\|a_{22}(\tau)\| \leq c\exp(-\kappa\tau) \quad \text{for} \quad \tau \geq 0 . \tag{26}$$

Lemma 4. The fundamental matrix $\Psi(\tau)$ for $d\delta_2/d\tau = a_{22}(\tau)\delta_2(\Psi(0) = E_k)$ satisfies:

(1) $\Psi(\infty) = \lim_{\tau \to \infty} \Psi(\tau)$ exists;

(2) $\det \Psi(\infty) \neq 0$;

(3) $\|\Psi(\tau) - \Psi(\infty)\| \leq c \exp(-\varkappa\tau)$.

Proof. Consider the matrix integral equation

$$\tilde{\Psi}(\tau) = E_k + \int_{\infty}^{\tau} a_{22}(s)\tilde{\Psi}(s)\,ds \; . \tag{27}$$

Applying successive approximations to it, we consider the sequence

$$\tilde{\Psi}_{n+1}(\tau) = E_k + \int_{\infty}^{\tau} a_{22}(s)\tilde{\Psi}_n(s)\,ds \; , \; \tilde{\Psi}_0(\tau) = E_k \; . \tag{28}$$

By virtue of (26) and (28),

$$\|\tilde{\Psi}_1(\tau) - \tilde{\Psi}_0(\tau)\| \leq \int_{\tau}^{\infty} \|a_{22}(s)\|ds \leq (c/\varkappa)\exp(-\varkappa\tau) \quad ,$$

$$\|\tilde{\Psi}_2(\tau) - \tilde{\Psi}_1(\tau)\| \leq \int_{\tau}^{\infty} \|a_{22}(s)\|\|\tilde{\Psi}_1(s) - \tilde{\Psi}_0(s)\|ds$$

$$\leq \frac{1}{2!}(c/\varkappa)^2\exp(-2\varkappa\tau) \quad ,$$

and

$$\|\tilde{\Psi}_n(\tau) - \tilde{\Psi}_{n-1}(\tau)\| \leq \frac{1}{n!}(c/\varkappa)^n\exp(-n\varkappa\tau)$$

$$\leq \frac{a^n}{n!} \quad \text{for} \quad a = c/\varkappa \; , \; n = 1,2,\ldots \quad .$$

Hence, it follows that

$$\tilde{\Psi}_n(\tau) = E_k + \sum_{i=1}^{n} [\tilde{\Psi}_i(\tau) - \tilde{\Psi}_{i-1}(\tau)]$$

converges as $n \to \infty$ (uniformly with respect to $\tau \geq 0$) to a matrix $\tilde{\Psi}(\tau)$ which satisfies equation (27) and consequently, $d\tilde{\Psi}/d\tau = a_{22}(\tau)\tilde{\Psi}$ and $\tilde{\Psi}(\infty) = E_k$. It follows also that $\det\tilde{\Psi}(0) \neq 0$, for otherwise $\det\tilde{\Psi}(\tau) \equiv 0$, contradicting the relation $\tilde{\Psi}(\infty) = E_k$. If we now set

$$\Psi(\tau) = \tilde{\Psi}(\tau)\tilde{\Psi}^{-1}(0) \quad ,$$

we obtain a fundamental matrix for which properties (1), (2) and (3) of Lemma 4 are satisfied.

E. Let us turn now to the matrix $R(\tau,\alpha)$ of Lemma 3.

<u>Lemma 5</u>. $\det R(\infty,\alpha) \neq 0$.

<u>Proof</u>. Denote the upper $m-k$ and the lower k rows of the matrix $B^{-1}(\alpha)\varphi_\alpha(0,\alpha)$ by h_1 and h_2 respectively, that is,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = B^{-1}(\alpha)\varphi_\alpha(0,\alpha) \quad . \tag{29}$$

Substituting $\varphi_\alpha(0,\alpha) = B(\alpha)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ into $F_x(\infty)\varphi_\alpha(0,\alpha) = 0$, multiplying on the left by $B^{-1}(\alpha)$ and taking account of (7), we obtain

$$\begin{pmatrix} C(\alpha) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0 \quad ;$$

whence, it follows that $C(\alpha)h_1 = 0$, so that $h_1 = 0$. Since the rank of $\varphi_\alpha(0,\alpha)$ is equal to k, $\det h_2 \neq 0$. From (29) we then obtain

$$h_2 = (B^{-1}(\alpha))_{21}[\varphi_\alpha(0,\alpha)]_1 + (B^{-1}(\alpha))_{22}[\varphi_\alpha(0,\alpha)]_2 \quad .$$

By virtue of (24) and (22),

$$h_2 = -(B^{-1}(\alpha))_{22}\{H(\infty)[\varphi_\alpha(0,\alpha)]_1 - [\varphi_\alpha(0,\alpha)]_2\}$$
$$= -(B^{-1}(\alpha))_{22}R(\infty,\alpha) \quad .$$

Since $\det(B^{-1}(\alpha))_{22} \neq 0$ (cf. (25)), $\det R(\infty,\alpha) \neq 0$.

§2 An Algorithm for the Construction of the Asymptotic Expansion of the Solution of the Initial Value Problem

The asymptotic expansion of the solution of problem (1), (2) will be constructed in the form

$$x(t,\mu) = \bar{x}(t,\mu) + \Pi x(\tau,\mu) \quad (\tau = t/\mu) \ , \tag{30}$$

where

$$\bar{x}(t,\mu) = \bar{x}_0(t) + \mu\bar{x}_1(t) + \dots + \mu^n\bar{x}_n(t) + \dots \quad ,$$
$$\Pi x(\tau,\mu) = \Pi_0 x(\tau) + \mu\Pi_1 x(\tau) + \dots + \mu^n\Pi_n x(\tau) + \dots \quad .$$

By substituting (30) into (1) and representing the function F in the form $F = \bar{F} + \Pi F$ just as in Chapter 1, we obtain a sequence of equations for the determination of $\bar{x}_i(t)$ and $\Pi_i x(\tau)$ $(i = 0,1,2,\dots)$.

For $\bar{x}_0(t)$ we have

$$F(\bar{x}_0(t),0,0) = 0 \quad .$$

By virtue of Condition I solutions of this equation can be written in the form

$$\bar{x}_0(t) = \varphi(t,\alpha(t)) \ , \tag{31}$$

where $\alpha(t)$ is an as yet arbitrary k-dimensional vector function.

For $\Pi_0 x(\tau)$ we obtain

$$\frac{d\Pi_0 x}{d\tau} = F(\varphi(0,\alpha(0)) + \Pi_0 x, 0, 0) \quad ,$$

which after the substitution $x = \Pi_0 x$ is made coincides exactly with equation (4) for $\alpha = \alpha(0)$. The initial condition for $\Pi_0 x(\tau)$ is obtained after substituting (30) into (2) as

$$\Pi_0 x(0) = x^0 - \varphi(0,\alpha(0)) = \begin{pmatrix} z^0 - \varphi_1(0,\alpha(0)) \\ y^0 - \varphi_2(0,\alpha(0)) \end{pmatrix} ,$$

where z^0 is the upper (m-k)-dimensional block of x^0, y^0 is the lower k-dimensional block, and φ_1, φ_2 are the analogous blocks of φ. Both the equation and the initial condition for $\Pi_0 x(\tau)$ then involve the as yet arbitrary vector $\alpha(0)$. Let us use this arbitrariness to guarantee the exponential convergence of $\Pi_0 x(\tau)$ to zero as $\tau \to \infty$. For this it is sufficient to require that $\Pi_0 x(0)$ belong to $\Omega(\alpha(0))$, that is, that $\Pi_0 x(0)$ satisfies (12). This gives

$$y^0 - \varphi_2(0,\alpha(0)) = P(z^0 - \varphi_1(0,\alpha(0)),\alpha(0)) \ . \tag{32}$$

Equation (32) is a k-dimensional vector equation for the k components of the vector $\alpha(0)$.

VI. Suppose that equation (32) has a solution $\alpha(0) = \alpha^0$.

Taking $\alpha(0) = \alpha^0$, $\Pi_0 x(\tau)$ belongs to $\Omega(\alpha^0)$ for $\tau \geq 0$, that is, the blocks $\Pi_0 z(\tau)$ and $\Pi_0 y(\tau)$ of $\Pi_0 x(\tau)$ satisfy $\Pi_0 y(\tau) = P(\Pi_0 z(\tau),\alpha^0)$ and consequently, for $\tau \geq 0$

$$\|\Pi_0 x(\tau)\| \le c\exp(-\varkappa\tau) \quad .$$

Remark. The functional determinant (corresponding to (32)) is

$$\Delta(\alpha) = \det\Big(-\frac{\partial\varphi_2(0,\alpha)}{\partial\alpha} + H(z^0 - \varphi_1(0,\alpha),\alpha)\frac{\partial\varphi_1(0,\alpha)}{\partial\alpha} -$$

$$P_\alpha(z^0 - \varphi_1(0,\alpha),\alpha)\Big) \quad .$$

If we denote $H(\Pi_0 z(\tau),\alpha)$ by $H(\tau)$, $H(\tau)[\varphi_\alpha(0,\alpha)]_1 - [\varphi_\alpha(0,\alpha)]_2$ by $R(\tau,\alpha)$ (as in §1), and $P(\Pi_0 z(\tau),\alpha)$ by $P(\tau,\alpha)$ then

$$\Delta(\alpha) = \det[R(0,\alpha) - P_\alpha(0,\alpha)] \quad .$$

For many singular perturbation situations analogous to this (cf., for example, [13, §13, Condition III]) it is assumed that the corresponding functional determinant is nonzero. In the present problem this requirement is unnecessary since it is not difficult to prove that $P_\alpha(\tau,\alpha) = R(\tau,\alpha) - \Psi(\tau)\Psi^{-1}(\infty)R(\infty,\alpha)$ ($\Psi(\tau)$ being the fundamental matrix from Lemma 4). Hence, it follows that $\Delta(\alpha) = \det\Psi^{-1}(\infty)R(\infty,\alpha) \neq 0$ by virtue of Lemmas 4 and 5.

Thus the function $\Pi_0 x(\tau)$ is completely determined, although $\alpha(t)$ occurs in the expression (31) for $\bar{x}_0(t)$. We only know $\alpha(0) = \alpha^0$. The function $\alpha(t)$ is determined completely from a solvability condition in the equation for $\bar{x}_1(t)$.

The equation for $\bar{x}_1(t)$ has the form

$$\frac{d\bar{x}_0}{dt} = F_x(\bar{x}_0(t),t,0)\bar{x}_1 + F_\mu(\bar{x}_0(t),t,0)$$

or

$$F_x(\varphi(t,\alpha(t)),t,0)\bar{x}_1 = \varphi_\alpha(t,\alpha(t))\frac{d\alpha}{dt} + \varphi_t(t,\alpha(t)) - F_\mu(\varphi(t,\alpha(t)),t,0) \quad . \tag{33}$$

The determinant of this linear algebraic system of equations is equal to zero. For the solvability of this system it is necessary and sufficient that the right-hand side be orthogonal to the eigenvectors $g_j(t,\alpha(t))(j = 1,\ldots,k)$ of the adjoint matrix $F_x^*(\varphi(t,\alpha(t)),t,0)$ corresponding to the eigenvalue $\lambda = 0$. Let us denote by $g(t,\alpha(t))$ the $(k \times m)$-matrix whose rows are the $g_j(t,\alpha(t))$. Then the orthogonality condition can be written as

$$\begin{aligned}&(g(t,\alpha(t))\varphi_\alpha(t,\alpha(t)))\frac{d\alpha}{dt} + \\ &(g(t,\alpha(t))[\varphi_t(t,\alpha(t)) - F_\mu(\varphi(t,\alpha(t)),t,0)]) = 0 \quad ,\end{aligned} \tag{34}$$

where $(g\varphi_\alpha)$ denotes the $(k \times k)$-product of g and φ_α ; analogous meaning is given to the other terms in (34). As noted in Chapter 1 $\det(g\varphi_\alpha) \neq 0$, and so (34) can be solved for $d\alpha/dt$:

$$\frac{d\alpha}{dt} = f_0(\alpha,t) \quad . \tag{35}$$

VII. Suppose that equation (35) together with the initial condition $\alpha(0) = \alpha^0$ has a solution $\alpha = \alpha(t)$ for t in $[0,T]$ that belongs to $D(\alpha)$ there, where $D(\alpha)$ is the domain in Condition II .

Through $\alpha(t)$ we completely determine the zero-th term of the approximation. Let us introduce the curve L consisting of the two pieces:

$$L_1 = \{(x,t): x = \bar{x}_0(0) + \pi_0 x(\tau)(\tau \geq 0) \; ; \; t = 0\} ,$$

$$L_2 = \{(x,t): x = \bar{x}_0(t); \; 0 \leq t \leq T\} .$$

It is natural to require

VIII. The curve L lies in the domain $D(x,t)$ of Condition I .

The solution of equation (33) can be written in the form

$$\bar{x}_1(t) = \bar{\varphi}_\alpha(t)\beta(t) + \tilde{\bar{x}}_1(t) , \tag{36}$$

where $\bar{\varphi}_\alpha(t) = \varphi_\alpha(t,\alpha(t)), \beta(t)$ is an as yet arbitrary k-dimensional vector function, and $\tilde{\bar{x}}_1(t)$ is a particular solution.

For $\pi_1 x(\tau)$,

$$\begin{aligned}\frac{d\pi_1 x}{d\tau} = F_x(\tau)\pi_1 x + [F_x(\tau) - \bar{F}_x(0)][\bar{x}_1(0) + \tau\bar{x}_0'(0)] + \\ [F_t(\tau) - \bar{F}_t(0)]\tau + [F_\mu(\tau) - \bar{F}_\mu(0)] ,\end{aligned} \tag{37}$$

where $F_x(\tau) = F_x(\bar{x}_0(0) + \pi_0 x(\tau),0,0)$, $\bar{F}_x(t) = F_x(\bar{x}_0(t),t,0)$, etc. Note that $F_x(\infty) = \bar{F}_x(0)$.

By using (36) for $\bar{x}_1(0)$ and since $\bar{F}_x(0)\bar{\varphi}_\alpha(0) = 0$,

$$\frac{d\pi_1 x}{d\tau} = F_x(\tau)\pi_1 x + F_x(\tau)\bar{\varphi}_\alpha(0)\beta(0) + \psi(\tau) , \tag{38}$$

where $\psi(\tau)$ is known and such that $\|\psi(\tau)\| \leq c\exp(-\varkappa\tau)$.

The initial condition is

$$\pi_1 x(0) = -\bar{x}_1(0) = -\bar{\varphi}_\alpha(0)\beta(0) - \tilde{x}_1(0) \quad .$$

Thus, an as yet arbitrary vector $\beta(0)$ appears in the equation and in the initial condition for $\pi_1 x(\tau)$. We use this arbitrariness to guarantee that $\pi_1 x(\tau)$ decreases exponentially as $\tau \to \infty$. Let us denote the upper and lower blocks of $\pi_1 x$ by $\pi_1 z$ and $\pi_1 y$, and let

$$\pi_1 z = \delta_1 \ , \ \pi_1 y = H(\tau)\delta_1 + \delta_2 \ ,$$

where $H(\tau) = H(\pi_0 z(\tau),\alpha)$. Lemma 2 implies the equations

$$\frac{d\delta_1}{d\tau} = a_{11}(\tau)\delta_1 + a_{12}(\tau)\delta_2 + [F_x(\tau)\bar{\varphi}_\alpha(0)]_1\beta(0) + \psi_1(\tau) \quad ,$$

$$\frac{d\delta_2}{d\tau} = a_{22}(\tau)\delta_2 + \{[F_x(\tau)\bar{\varphi}_\alpha(0)]_2 - H(\tau)[F_x(\tau)\bar{\varphi}_\alpha(0)]_1\}\beta(0)$$

$$+ [\Psi_2(\tau) - H(\tau)\psi_1(\tau)] \tag{39}$$

with initial conditions

$$\delta_1(0) = -[\bar{\varphi}_\alpha(0)]_1\beta(0) - \tilde{z}_1(0) \quad ,$$

$$\delta_2(0) = \{H(0)[\bar{\varphi}_\alpha(0)]_1 - [\bar{\varphi}_\alpha(0)]_2\}\beta(0) + \{H(0)\tilde{z}_1(0) - \tilde{y}_1(0)\} \ . \tag{40}$$

Using Lemma 3 and introducing $\delta_2^0 = \{H(0)\tilde{z}_1(0) - \tilde{y}_1(0)\}$ we have

$$\delta_2(\tau) = R(\tau,\alpha^0)\beta(0) + \Psi(\tau)\delta_2^0 + \int_0^\tau \Psi(\tau)\Psi^{-1}(s)[\psi_2(s) - H(s)\psi_1(s)]ds \quad . \tag{41}$$

By requiring that $\delta_2(\tau) \to 0$ as $\tau \to \infty$ we obtain

$$R(\infty,\alpha^0)\beta(0) = -\Psi(\infty)\{\delta_2^0 + \int_0^\infty \Psi^{-1}(s)[\psi_2(s) - H(s)\psi_1(s)]ds\} \quad . \tag{42}$$

By virtue of Lemma 5 this equation is uniquely solvable for $\beta \overset{0}{=} \beta(0)$. Substituting into (41) and using the exponential decay of $\Psi(\tau)$ and $H(\tau)$ we have

$$||\delta_2(\tau)|| \leq c\exp(-\kappa\tau) \quad \text{for} \quad \tau \geq 0 \quad . \tag{43}$$

Since $F_x(\tau)\bar{\varphi}_\alpha(0) = [F_x(\tau) - \bar{F}_x(0)]\bar{\varphi}_\alpha(0)$ satisfies the same exponential estimate ,

$$\frac{d\delta_1}{d\tau} = a_{11}(\tau)\delta_1 + \tilde{\psi}_1(\tau) \quad ,$$

where $||\tilde{\psi}_1(\tau)|| \leq c\exp(-\kappa\tau)$. Thus

$$\delta_1(\tau) = \Phi(\tau)\delta_1(0) + \int_0^\tau \Phi(\tau)\Phi^{-1}(s)\tilde{\psi}_1(s)ds \quad ,$$

where the fundamental matrix $\Phi(\tau)$ satisfies (23), and so

$$||\delta_1(\tau)|| \leq c\exp(-\kappa\tau) + \int_0^\tau c\exp(-\kappa(\tau-s))c\exp(-\kappa s)ds$$

$$\leq c\exp(-\kappa\tau) \quad . \tag{44}$$

$$\frac{d\bar{x}_1}{dt} = \bar{F}_x(t)\ \bar{x}_2 + \frac{1}{2}(\bar{x}_1,\ \bar{F}_{xx}(t)\bar{x}_1) + \bar{F}_{x\mu}(t)\ \bar{x}_1 + \frac{1}{2}\bar{F}_{\mu\mu}(t). \tag{45}$$

Here $(\bar{x}_1, \bar{F}_{xx}(t)\bar{x}_1)$ is the vector whose components are the scalar products $\langle\bar{x}_1, \bar{F}^{\ell}_{xx}\ \bar{x}_1\rangle = \sum_{i,j=1}^{m} (\partial^2\bar{F}^{\ell}/\partial x^i \partial x^j)(t)\bar{x}^i\bar{x}^j$ ($\ell=1,\ldots,m$). Substituting $\bar{x}_1$ from formula (36) and writing the solvability condition for equation (45) in a form analogous to the condition (34), we obtain the equation

$$\frac{d\beta}{dt} = f_1(\beta,t). \tag{46}$$

At first glance it may seem that $f_1(\beta,t)$ depends quadratically on $\beta(t)$ as a result of the term $(\bar{x}_1,\bar{F}_{xx}(t)\bar{x}_1)$ in (45). However this is not the case, since

$$< g_j(t,\alpha(t)),\ (\bar{\phi}_\alpha(t)\ \beta(t),\ \bar{F}_{xx}(t)\ \bar{\phi}_\alpha(t)\ \beta(t)) > = 0 \tag{47}$$

for $j = 1,\ldots,k$. In order to verify (47) we differentiate the identity $F(\phi(t,\alpha),t,0) = 0$ twice with respect to the components α_p and α_q of the vector $\alpha (p,q = 1,\ldots,k)$ and obtain

$$\left(\frac{\partial\phi}{\partial\alpha_p},\ F_{xx}\frac{\partial\phi}{\partial\alpha_q}\right) + F_x\frac{\partial^2\phi}{\partial\alpha_p\partial\alpha_q} = 0.$$

Then, forming the scalar product with $g_j(t,\alpha)$ and noting that $< g_j,\ F_x \frac{\partial^2\phi}{\partial\alpha_p\partial\alpha_q} > = 0$ since $F_x{}^*g_j = 0$, we see that

$$< g_j,\ \left(\frac{\partial\phi}{\partial\alpha_p},\ F_{xx}\frac{\partial\phi}{\partial\alpha_q}\right) > = 0$$

for j, p, q = 1,...,k.

Thus equation (46) is linear, that is, $f_1(\beta,t) = A(t)\ \beta(t) + B_1(t)$, where expressions for $A(t)$ and $B_1(t)$ are obtained easily from (45). By virtue of this linearity, there is a unique solution of (46) which exists in [0,T] and satisfies $\beta(0) = \beta_0$.

Thus the term of order μ in the asymptotic expansion is completely determined. The determination of the successive terms of the expansion proceeds in a manner analogous to that of $\bar{x}_1(t)$ and $\Pi_1 x(\tau)$. At the i-th step an arbitrary function (say $\gamma(t)$) appears in the expression for $\bar{x}_i(t)$. First we determine $\gamma(0)$ from the condition that $\Pi_i\ x(\tau) \to 0$ as $\tau \to \infty$; the equation for $\gamma(0)$ is of the same type as (42), with det $R(\infty,\alpha^0) \neq 0$. Then from the solvability condition in the equation for $\bar{x}_{i+1}(t)$ we obtain an equation for $\gamma(t)$ like (46), namely

$$\frac{d\gamma}{dt} = A(t)\ \gamma + B_i(t),$$

which determines $\gamma(t)$ uniquely in [0,T].

Thus it is possible, under Conditions I-VIII, to construct arbitrarily many terms of the series (30).

§3 An Estimate of the Remainder Term

As in Chapter 1 let us first make a more precise formulation of Condition I concerning the smoothness of the function $F(x,t,\mu)$. We note that it is possible to take an arbitrary δ-tube of the curve L (cf. VIII) to be the domain $D(x,t)$.

I. Suppose that the function $F(x,t,\mu)$ has continuous partial derivatives with respect to each argument up to order $(n+2)$ inclusive in the domain $D(x,t,\mu) = D(x,t) \times [0,\mu_0]$.

Let us set

$$X_k(t,\mu) = \sum_{i=0}^{k} \mu^i(\bar{x}_i(t) + \Pi_i x(\tau)) .$$

<u>Theorem 2</u>. Under Conditions I-VIII there exist positive constants μ_0 and c such that for $0 < \mu \leq \mu_0$ the solution $x(t,\mu)$ of the problem (1), (2) exists in the interval $[0,T]$, is unique and satisfies the inequality

$$\|x(t,\mu) - X_n(t,\mu)\| \leq c\mu^{n+1} \quad (0 \leq t \leq T) .$$

<u>Proof</u>. Substituting $x = X_{n+1} + \xi$ into equation (1) we obtain an equation

$$\mu\frac{d\xi}{dt} = F_x(t,\mu)\xi + G(\xi,t,\mu) , \tag{48}$$

where

$$F_x(t,\mu) = F_x(X_1(t,\mu),t,\mu)$$

and

$$G(\xi,t,\mu) = F(X_{n+1}(t,\mu) + \xi,t,\mu) - F_x(t,\mu)\xi - \mu\frac{dX_{n+1}(t,\mu)}{dt} .$$

The function $G(\xi,t,\mu)$ has the following two important properties which can be established just as easily as in [13, §10, Subsection 4], namely

1. $G(0,t,\mu) = \mathcal{O}(\mu^{n+2})$.

2. If $\|\xi_1(t,\mu)\| \leq c_1\mu^2$ and $\|\xi_2(t,\mu)\| \leq c_1\mu^2$ for $0 \leq t \leq T$ and $0 < \mu \leq \mu_1$ (for some constants c_1 and μ_1), then there exist constants c_0 and $\mu_0 \leq \mu_1$ such that for $0 \leq t \leq T$ and $0 < \mu \leq \mu_0$

$$\|G(\xi_1,t,\mu) - G(\xi_2,t,\mu)\| \leq c_0\mu^2 \max_{[0,T]} \|\xi_1 - \xi_2\| \tag{49}$$

(for μ_0 note the remark in Subsection 3, §1, Chapter 1) . In conformity with Chapter 1 $G(\xi,t,\mu)$ is a contraction operator with contraction coefficient of order $\mathcal{O}(\mu^2)$ for $\xi = \mathcal{O}(\mu^2)$.

We now introduce the change of variables

$$\xi = T(t)\binom{u}{v} ,$$

where

$$T^{-1}(t)\bar{F}_x(t)T(t) = \begin{pmatrix} \bar{a}_{11}(t) & 0 \\ 0 & 0 \end{pmatrix} ,$$

for $\bar{F}_x(t) = F_x(\bar{x}_0(t),t,0)$. Here the eigenvalues $\lambda_i(t,\alpha(t))$ of the $((m-k) \times (m-k))$-matrix $a_{11}(t)$ satisfy condition (3) . Then

$$T^{-1}(t)F_x(t,\mu)T(t) = \begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{pmatrix} ,$$

so

$$\|a_{11}(t,\mu) - \bar{a}_{11}(t)\| \leq c(\exp(-\varkappa t/\mu) + \mu) ,$$

while the other blocks $a_{ik}(t,\mu)$ satisfy

$$\|a_{ik}(t,\mu)\| \leq c(\exp(-\varkappa t/\mu) + \mu) .$$

The system for u and v has the form

$$\begin{aligned} \mu\frac{du}{dt} &= a_{11}(t,\mu)u + a_{12}(t,\mu)v - \mu b_{11}(t)u - \mu b_{12}(t)v + (T^{-1}G)_1 , \\ \mu\frac{dv}{dt} &= a_{21}(t,\mu)u + a_{22}(t,\mu)v - \mu b_{21}(t)u - \mu b_{22}(t)v + (T^{-1}G)_2 , \end{aligned} \tag{50}$$

where the $b_{ik}(t)$ are the blocks of $T^{-1}(t)\frac{dT(t)}{dt}$. Note that $A_{ik}(t,\mu) = a_{ik}(t,\mu) - \mu b_{ik}(t)$ satisfy the same inequalities as the $a_{ik}(t,\mu)$.

Suppose now that $U(t,s,\mu)$ and $V(t,s,\mu)$ are the fundamental matrices of the homogeneous systems

$$\mu\frac{du}{dt} = A_{11}(t,\mu)u \quad (U(s,s,\mu) = E_{m-k}) ,$$

$$\mu\frac{dv}{dt} = A_{22}(t,\mu)v \quad (V(s,s,\mu) = E_k) .$$

By the properties of $\bar{a}_{11}(t)$, $A_{11}(t,\mu)$ and $A_{22}(t,\mu)$, these fundamental matrices satisfy for $0 \leq s \leq t \leq T$ and $0 < \mu \leq \mu_0$

$$\|U(t,s,\mu)\| \leq c\exp(-\kappa(t-s)/\mu),\ \|V(t,s,\mu)\| \leq c \quad .$$

Using the fundamental matrix $V(t,s,\mu)$ and the trivial initial values of u and v (as well as of ξ), we can express the second equation in (50) as the integral equation

$$v(t,\mu) = \int_0^t K_2(t,s,\mu)u(s,\mu)ds + Q_2(u,v,t,\mu) \quad . \tag{51}$$

Here the kernel

$$K_2(t,s,\mu) = \mu^{-1}V(t,s,\mu)A_{21}(s,\mu)$$

clearly satisfies the inequality

$$\|K_2(t,s,\mu)\| \leq c[\mu^{-1}\exp(-\kappa s/\mu) + 1] \quad , \tag{52}$$

while the integral operator

$$Q_2(u,v,t,\mu) = \mu^{-1}\int_0^t V(t,s,\mu)(T^{-1}G)_2 ds$$

by the two properties of $G(\xi,t,\mu)$ satisfies the estimate $Q_2(0,0,t,\mu) = O(\mu^{n+1})$ and is a contraction operator with an order $O(\mu)$ contraction coefficient for u and v of order $O(\mu^2)$.

By substituting (51) into the first equation of (50) and using $U(t,s,\mu)$, we obtain the integral equation

$$u(t,\mu) = \int_0^t K_1(t,s,\mu)u(s,\mu)ds + Q_1(u,v,t,\mu) \quad , \tag{53}$$

where the kernel

$$K_1(t,s,\mu) = \mu^{-1}\int_s^t U(t,p,\mu)A_{12}(p,\mu)K_2(p,s,\mu)dp$$

satisfies the same inequality as $K_2(t,s,\mu)$ (cf. (52)), while the integral operator

$$Q_1(u,v,t,\mu) = \mu^{-1}\int_0^t U(t,s,\mu)[A_{12}(s,\mu)Q_2(u,v,s,\mu) + (T^{-1}G)_1]ds$$

has the same two properties as $Q_2(u,v,t,\mu)$.

Let us denote by $R(t,s,\mu)$ the resolvent kernel of $K_1(t,s,\mu)$. It satisfies the same estimates as the kernel itself. We can express (53) as the equivalent equation

$$\begin{aligned} u(t,\mu) &= Q_1(u,v,t,\mu) + \int_0^t R(t,s,\mu)Q_1(u,v,s,\mu)ds \\ &\equiv S_1(u,v,t,\mu) \quad , \end{aligned}$$

where the integral operator $S_1(u,v,t,\mu)$ has the same two properties as $Q_1(u,v,t,\mu)$.

Substituting (54) into (51) we obtain

$$\begin{aligned} v(t,\mu) &= \int_0^t K_2(t,s,\mu)S_1(u,v,s,\mu)ds + Q_2(u,v,t,\mu) \\ &\equiv S_2(u,v,t,\mu) \quad , \end{aligned}$$

where S_2 has the properties of S_1. Therefore we can apply the method of successive approximations to the system (54), (55) (with $u_0 = v_0 \equiv 0$ and easily show as in [13, §10] that for sufficiently small μ a solution $u(t,\mu)$, $v(t,\mu)$ exists in the interval $[0,T]$, is unique and satisfies the estimates $u(t,\mu) = \Theta(\mu^{n+1})$ and $v(t,\mu) = \Theta(\mu^{n+1})$. Hence, it follows directly that $\xi(t,\mu) = x(t,\mu) - X_{n+1}(t,\mu) = \Theta(\mu^{n+1})$, so $x(t,\mu) - X_n(t,\mu) = \Theta(\mu^{n+1})$ and this proves the theorem.

§4 Special Cases

1. Consider the system of equations

$$\begin{aligned} \mu\frac{dz}{dt} &= A(y,t)z + B(y,t) \\ & \qquad\qquad\qquad (0 \le t \le T) \\ \frac{dy}{dt} &= C(y,t)z + D(y,t) \end{aligned} \tag{56}$$

with the infinitely large (as $\mu \to 0$) initial condition

$$z(0,\mu) = z^0/\mu\ , \ y(0,\mu) = y^0\ . \tag{57}$$

In the special case that z and y are scalar functions with $C(y,t) \equiv 1$ and $D(y,t) \equiv 0$ this problem was considered in detail in [13, §16].

Suppose now that in the system (56) z is an $(m-k)$-dimensional vector and y a k-dimensional vector. Let us introduce in place of z the function μz (which we will again denote by z), then (56), (57) takes the form

$$\mu \frac{dz}{dt} = A(y,t)z + \mu B(y,t) ,$$

$$\mu \frac{dy}{dt} = C(y,t)z + \mu D(y,t) ,$$

$$z(0,\mu) = z^0 , \ y(0,\mu) = y^0 .$$

This is a problem of the form (1), (2) for the m-dimensional vector function $x = \binom{z}{y}$. The reduced system

$$A(\bar{y},t)\bar{z} = 0 , \ C(\bar{y},t)\bar{z} = 0$$

has the family of solutions

$$\bar{z} = 0 , \ \bar{y} = \alpha \equiv \varphi(t,\alpha) .$$

The matrix $F_x(\varphi(t,\alpha),t,0)$ can be written in block form as

$$\begin{pmatrix} A(\alpha,t) & 0 \\ C(\alpha,t) & 0 \end{pmatrix}$$

and, consequently, Condition III is satisfied provided the eigenvalues $\lambda_i(t,\alpha)$ $(i = 1,\ldots k)$ of the matrix $A(\alpha,t)$ satisfy the inequality (3), that is,

$$\operatorname{Re} \lambda_i(t,\alpha) < 0 . \tag{58}$$

The matrix $\varphi_\alpha(t,\alpha)$, consisting of the eigenvectors corresponding to $\lambda \equiv 0$, now has the form

$$\varphi_\alpha(t,\alpha) = \begin{pmatrix} 0 \\ E_k \end{pmatrix} ,$$

and the system (4) can be written as

$$\frac{dz}{d\tau} = A(\alpha+y,0)z \ , \ \frac{dy}{d\tau} = C(\alpha+y,0)z \quad . \tag{59}$$

Suppose that z and y are scalars, that is, $m-k = k = 1$. Then condition (58) reduces to $A(y,t) < 0$. If we assume that $C(y,t)$ is of constant sign then from (59) we obtain an explicit representation of the manifold $\Omega(\alpha)$, namely

$$z = \int_0^y \frac{A(\alpha+y,0)}{C(\alpha+y,0)}\, dy \ , \tag{60}$$

which agrees exactly with the formula in (12) .

If $C(y,t) \equiv 1$ and $D(y,t) \equiv 0$ (this case was discussed in [13, §16]), then $\varphi_\alpha = \binom{0}{1}$, $g = (1,-A)$, $F_\mu = \binom{B}{0}$, and equation (34) assumes the form

$$A(\alpha,t)\frac{d\alpha}{dt} + B(\alpha,t) = 0 \quad .$$

This agrees precisely with equation (4.385) of [13,§16] , with $\alpha(t)$ playing the role of $\overline{y}_0(t)$. The initial condition for $\alpha(t)$ is determined from equation (32) which, in the present case, through (60) can be written as

$$z^0 = \int_0^{y^0-\alpha(0)} A(\alpha(0) + y,0)\,dy \quad ;$$

hence, we obtain

$$z^0 = \int_{\alpha(0)}^{y^0} A(\eta,0)\,d\eta \quad \text{or} \quad z^0 + \int_{y^0}^{\alpha(0)} A(\eta,0)\,d\eta = 0 \quad .$$

The succeeding equations agree exactly with the formulation in equation (4.395) of [13, §16] , and from them we can determine $\bar{y}_0(0)$. Thus α coincides with $\bar{y}_0(0)$, that is, the formal construction of §2 reduces to the results obtained in [13, §16] .

2. The general singularly perturbed initial value problem

$$\mu \frac{dz}{dt} = F(z,y,t) \ , \ \frac{dy}{dt} = f(z,y,t) \ (0 \le t \le T) \tag{61}$$

$$z(0,\mu) = z^0 \ , \ y(0,\mu) = y^0 \ , \tag{62}$$

which was considered in detail in [13], can be reduced to a problem of the form (1), (2) . To accomplish this we multiply the second equation by μ , and set $x = \begin{pmatrix} z \\ y \end{pmatrix}$ and $G(x,t,\mu) = \begin{pmatrix} F \\ \mu f \end{pmatrix}$. We obtain

$$\mu \frac{dx}{dt} = G(x,t,\mu) \quad , \tag{63}$$

$$x(0,\mu) = x^0 = \begin{pmatrix} z^0 \\ y^0 \end{pmatrix} , \tag{64}$$

whose reduced system has the family of solutions

$$\bar{y} = \alpha \ , \ \bar{z} = \varphi(t,\alpha) \quad ,$$

where $\varphi(t,\alpha)$ is a z-root of the equation $F(z,\alpha,t) = 0$. It is possible to develop the construction of the asymptotic solution of the problem (63), (64) by the scheme of §2 , which after a finite number of calculations gives the same result as in [13, Chapter 3] .

§5 Applications of the Asymptotic Method to Problems in Kinetics

1. The Equations of Chemical Kinetics. Suppose that there are n chemical reactions involving m substances, namely

$$\alpha_{i1}X_1 + \dots + \alpha_{im}X_m \underset{k_i^-}{\overset{k_i^+}{\rightleftharpoons}} \beta_{i1}X_1 + \dots + \beta_{im}X_m$$

$$(i = 1,\dots,n) \quad .$$

Here X_i denotes the i-th substance, α_{ik}, β_{ik} are integers denoting the number of molecules of the k-th substance which participate in the i-th reaction (corresponding to the forward and reverse reaction, respectively), and k_i^+, k_i^- are the rate constants of these reactions.

If we denote the concentration of the ℓ-th substance by x_ℓ, then the changes in x_ℓ during the time dt, determined by the reaction rates k_i^+, k_i^-, are given respectively by

$$dx_\ell = -k_i^+ x_1^{\alpha_{i1}} \dots x_m^{\alpha_{im}}(\alpha_{i\ell}-\beta_{i\ell})dt \quad ,$$

$$dx_\ell = k_i^- x_1^{\beta_{i1}} \dots x_m^{\beta_{im}}(\alpha_{i\ell}-\beta_{i\ell})dt \quad ,$$

and consequently, the total change in x_ℓ (as a result of all the reactions) is equal to

$$dx_\ell = \sum_{i=1}^{n} \gamma_{i\ell} w_i dt \ , \text{ where } \ w_i = k_i^+ x_1^{\alpha_{i1}} \dots x_m^{\alpha_{im}} - k_i^- x_1^{\beta_{i1}} \dots x_m^{\beta_{im}}$$

and $\gamma_{i\ell} = \beta_{i\ell} - \alpha_{i\ell}$.

Thus we are led to the system of differential equations

$$\frac{dx_\ell}{dt} = \sum_{i=1}^{n} \gamma_{i\ell} w_i \quad (\ell = 1,\ldots,m) \ . \tag{65}$$

Under actual conditions the rate constants differ widely from each other. This property can be expressed by means of a small parameter μ. Suppose then that $k_i^{\pm} = \mu^{-1} \overline{k_i^{\pm}}$ $(i = 1,\ldots, \bar{n} < n)$. Then we have

$$\bar{w}_i = \mu w_i = \bar{k}_i^{+} x_1^{\alpha_{i1}} \ldots x_m^{\alpha_{im}} - \bar{k}_i^{-} x_1^{\beta_{i1}} \ldots x_m^{\beta_{im}} \quad (i = 1,\ldots,\bar{n}) \ .$$

So the system (65) can be written as

$$\mu \frac{dx_\ell}{dt} = \sum_{i=1}^{\bar{n}} \gamma_{i\ell} \bar{w}_i + \mu \sum_{i=\bar{n}+1}^{n} \gamma_{i\ell} w_i \quad (\ell = 1,\ldots,m) \ . \tag{66}$$

Setting $\mu = 0$ we obtain the reduced system

$$0 = \sum_{i=1}^{\bar{n}} \gamma_{i\ell} \bar{w}_i \quad (\ell = 1,\ldots,m) \ . \tag{67}$$

In practice it often happens that system (67) has a family of solutions which depend on one or more arbitrary parameters, and thus, the problem reduces to a singularly perturbed equation (66) in the critical case.

One method for determining approximate solutions of the equations of chemical kinetics containing a small parameter is known in physical chemistry as the method of quasi-stationary concentrations of Semenov-Bodenstein. A number of works are devoted to questions involving the mathematical justification of this method (that is, to a justification of the passage to the limit as $\mu \to 0$); cf., for example, [17,18].

We now discuss an example of an actual chemical reaction for which the calculations can be carried out using the asymptotic methods presented in this chapter. This system is

$$\frac{dx_1}{dt} = -k_1^+x_1 + k_1^-x_2 \ , \ \frac{dx_2}{dt} = k_1^+x_1 - k_1^-x_2 - k_3^+x_2x_3 - k_4^+x_2x_4 \ ,$$
$$\frac{dx_3}{dt} = -k_3^+x_2x_3 \ , \ \frac{dx_4}{dt} = -k_4^+x_2x_4 \ . \qquad (68)$$

[Such a system occurs in investigations of the reaction kinetics of organometallic compounds and was proposed by A.N. Kashinym, a colleague of ours in the chemistry department of Moscow State University.] The rate constants have the orders of magnitude

$$k_1^+ \sim 10 \ , \ k_1^- \sim 10^9 \ , \ k_3^+ \sim 10^{10} \ , \ k_4^+ \sim 10^8 \ , \ k_3^- = k_4^- = 0 \ .$$

Dividing each of these equations by k_4^+ and making the substitutions $\mu = 1/k_4^+$, $a = k_1^+$, $b = k_1^-/k_4^+$ and $k_3^+/k_4^+ = c$, we obtain

$$\mu\frac{dx_1}{dt} = -\mu\, ax_1 + bx_2 \ , \ \mu\frac{dx_2}{dt} = \mu\, ax_1 - bx_2 - cx_2x_3 - x_2x_4 \ ,$$
$$\mu\frac{dx_3}{dt} = -cx_2x_3 \ , \ \mu\frac{dx_4}{dt} = -x_2x_4 \ . \qquad (69)$$

The reduced system

$$0 = b\bar{x}_2 \ , \ 0 = -b\bar{x}_2 - c\,\bar{x}_2\bar{x}_3 - \bar{x}_2\bar{x}_4 \ ,$$
$$0 = -c\,\bar{x}_2\bar{x}_3 \ , \quad 0 = -\bar{x}_2\bar{x}_4$$

has a family of solutions depending on three arbitrary parameters, namely

$$\bar{x}_1 = \alpha_1 \ , \ \bar{x}_2 = 0 \ , \ \bar{x}_3 = \alpha_2 \ , \ \bar{x}_4 = \alpha_3 \ . \tag{70}$$

Here the matrix $F_x(\varphi(t,\alpha),t,0)$ is

$$\begin{pmatrix} 0 & b & 0 & 0 \\ 0 & -b - c\alpha_2 - \alpha_3 & 0 & 0 \\ 0 & -c\alpha_2 & 0 & 0 \\ 0 & -\alpha_3 & 0 & 0 \end{pmatrix} ,$$

and its eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = -b - c\alpha_2 - \alpha_3$. Since $b > 0$, $c > 0$ and α_2 , α_3 are nonnegative (which makes sense physically), it follows that $\lambda_4 < 0$. Thus Conditions I-III of §1 are satisfied.

The system (4) now has the form

$$\frac{dx_1}{d\tau} = bx_2 \ , \ \frac{dx_2}{d\tau} = -bx_2 - cx_2(\alpha_2+x_3) - x_2(\alpha_3+x_4) \ ,$$

$$\frac{dx_3}{d\tau} = -cx_2(\alpha_2+x_3) \ , \ \frac{dx_4}{d\tau} = - x_2(\alpha_3+x_4) \ .$$

This system is sufficiently simple that it can be integrated, and we obtain for the manifold $\Omega(\alpha)$ the exact representation

$$\begin{aligned} x_2 &= - x_1 + (\exp(-\tfrac{c}{b}x_1) - 1)\alpha_2 + (\exp(-\tfrac{x_1}{b}) - 1)\alpha_3 \ , \\ x_3 &= (\exp(-\tfrac{c}{b}x_1) - 1)\alpha_2 \ , \ x_4 = (\exp(-\tfrac{x_1}{b}) - 1)\alpha_3 \ . \end{aligned} \tag{71}$$

Thus, as in (12), the lower block (consisting of three components) of the vector x is expressed in terms of the upper block (consisting of one component). By the same token Condition V is satisfied. An elementary argument verifies Condition IV .

Suppose now that the system (69) is furnished with the initial condition $x(0,\mu) = x^0$. Then the vector equation (32) assumes the form

$$\begin{aligned} x_2^0 &= -(x_1^0 - \alpha_1(0)) + \{\exp[-\tfrac{c}{b}(x_1^0 - \alpha_1(0))] - 1\}\alpha_2(0) \\ &\quad + \{\exp[-\tfrac{1}{b}(x_1^0 - \alpha_1(0))] - 1\}\alpha_3(0) \quad , \\ x_3^0 - \alpha_2(0) &= \{\exp[-\tfrac{c}{b}(x_1^0 - \alpha_1(0))] - 1\}\alpha_2(0) \quad , \\ x_4^0 - \alpha_3(0) &= \{\exp[-\tfrac{1}{b}(x_1^0 - \alpha_1(0))] - 1\}\alpha_3(0) \quad . \end{aligned} \tag{72}$$

By setting $t = x_1^0 - \alpha_1(0)$ we can determine $\alpha_2(0)$ and $\alpha_3(0)$ in terms of t , namely

$$\alpha_2(0) = x_3^0 \exp(+\tfrac{c}{b}t) \ , \ \alpha_3(0) = x_4^0 \exp(+\tfrac{1}{b}t) \quad .$$

Substituting into the first equation of (72) we obtain an equation for t

$$t + x_3^0 \exp(+\tfrac{c}{b}t) + x_4^0 \exp(+\tfrac{1}{b}t) = x_3^0 + x_4^0 - x_2^0 \quad .$$

Elementary considerations show that this equation has a unique solution for $x_3^0 \geq 0$ and $x_4^0 \geq 0$. Thus $\alpha_1(0)$, $\alpha_2(0)$ and $\alpha_3(0)$ are uniquely determined from (72), that is, Condition VI holds.

One can also write equations for $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$. In the present case it is a matter of integrating by quadratures. Thus one can determine $\bar{x}_{10}(t) = \alpha_1(t)$, $\bar{x}_{20}(t) = 0$, $\bar{x}_{30}(t) = \alpha_2(t)$ and $\bar{x}_{40}(t) = \alpha_3(t)$.

The determination of $\Pi_0 x(\tau)$ reduces to the integration of the scalar equation

$$\frac{d\pi_0 x_1}{d\tau} = b\{-\pi_0 x_1 + [\exp(-\frac{c}{b}\pi_0 x_1) - 1]\alpha_2(0) + [\exp(-\frac{1}{b}\pi_0 x_1) - 1)\alpha_3(0)\} \tag{73}$$

with $\pi_0 x_1(0) = x_1^0 - \alpha_1(0)$ by quadratures. After determining $\pi_0 x_1(\tau)$ the remaining functions $\pi_0 x_i(\tau)$ $(i = 2,3,4)$ are found by means of the equation for $\Omega(\alpha)$ (cf. (71)) .

Using the scheme of §2 we can also construct the successive terms of the asymptotic expansion.

2. Equations of a Nonequilibrium Gas. The following equations are valid for a spatially homogeneous gas with a distribution of velocities at equilibrium:

$$\mu\frac{dn_i}{dt} = \Sigma_{i0}(n,T) + \mu\,\Sigma_{i1}(n,T) \; (i = 1,\ldots,N) \; , \qquad \frac{dT}{dt} = -\frac{2}{3}\Sigma_2(n,T) \; . \tag{74}$$

Here n_i $(i = 1,\ldots,N)$ denotes the density of those particles with internal energy e_i and T is the translational temperature [25] . To these are added certain conditions $n_i(0,\mu) = n_i^0$ and $T(0,\mu) = T^0$. $\Sigma_{i0}(n,T)$ characterizes the change in n_i as a result of exchanges of energy in collisions, while $\Sigma_{i1}(n,T)$ characterizes the change in n_i as a result of the transfer of internal energy to the energy of translational motion. The small parameter μ signifies that the transfer of internal energy to translational energy is considerably less likely than the exchange of internal energy as a result of collisions. For particles of equal mass we have

$$\Sigma_{i0}(n,T) = \sum_{\substack{k,\ell,m \\ \Delta e=0}} Q_{k\ell}^{im}(T)\ (n_k n_\ell - n_i n_m) \quad ,$$

$$\Sigma_{i1}(n,T) = \sum_{\substack{k,\ell,m \\ \Delta e\neq 0}} P_{k\ell}^{im}(T) n_k n_\ell - P_{im}^{k\ell}(T) n_i n_m) \quad ,$$

and

$$\Sigma_2(n,T) = \sum_{k,\ell,m,i} e_i (P_{k\ell}^{im}(T) n_k n_\ell - P_{im}^{k\ell}(T) n_i n_m) \quad .$$

Here $\Delta e = e_i + e_m - e_k - e_\ell$, while $Q_{k\ell}^{im}(T)$ and $P_{k\ell}^{im}(T)$ are the probability of the exchange of internal energy in collisions and the probability of the transfer of internal energy to translational energy.

The reduced equation $\Sigma_{i0}(\overline{n}, \overline{T}) = 0$ has solutions obtained from the condition that $\overline{n}_k \overline{n}_\ell = \overline{n}_i \overline{n}_m$; whence, taking note of the fact that $\Delta e = e_i + e_m - e_k - e_\ell = 0$ we have that $n_k = \alpha \exp(\beta e_k)$ for arbitrary parameters α and β . The Boltzmann distribution, in which $-1/\beta$ denotes the internal temperature and which depends on t , is found by means of the following approximations which agree with the general rules stated above in §2 .

The system of equations for the π_0-functions has the form

$$\frac{d\pi_0 n_i}{d\tau} = \Sigma_{i0}(\overline{n}(0) + \pi_0 n,\ T(0)) \quad (i = 1,\ldots,N) \quad .$$

The law of conservation of particles and the law of conservation of energy can themselves be represented by two first integrals of this system, namely

$$\sum_i (\bar{n}_i(0) + \pi_0 n_i) = \text{const.} = \sum_i \bar{n}_i(0)$$

and

$$\sum_i e_i(\bar{n}_i(0) + \pi_0 n_i) = \text{const.} = \sum_i e_i \bar{n}_i(0) \quad ,$$

and consequently, the equations

$$\sum_i \pi_0 n_i = 0 \ , \ \sum_i e_i \pi_0 n_i = 0$$

furnish a 2-dimensional stability manifold, while the equations

$$\sum_i [n_i^0 - \alpha(0)\exp(\beta(0)e_i)] = 0 \ , \ \sum_i e_i [n_i^0 - \alpha(0)\exp(\beta(0)e_i)] = 0$$

lead to a determination of $\alpha(0)$ and $\beta(0)$. We note that $\sum_i n_i^0 = 1$ implies immediately that $\alpha(0)$ can be determined in terms of $\beta(0)$, that is, $\alpha(0) = 1/\sum \exp(\beta(0)e_i)$, after which $\beta(0)$ can be determined from the second equation.

Equations for $\alpha(t)$ and $\beta(t)$ can be obtained from the general rule of §2 involving orthogonality conditions. In the present case it is clear that system (74) has a first integral of the form

$$T + \frac{2}{3}\sum_i e_i n_i = \text{const.} = T^0 + \frac{2}{3}\sum_i e_i n_i^0 \ , \ \sum_i n_i = 1 \quad .$$

Whence, by virtue of the fact that the π_0-function converges to zero as $\tau \to \infty$, we have that

$$\bar{T}(t) + \frac{2}{3}\sum_i e_i \bar{n}_i(t) = T^0 + \frac{2}{3}\sum_i e_i n_i^0 \ , \ \sum_i \bar{n}_i(t) = 1 \quad . \tag{75}$$

From the second equation in (75) $\alpha(t) = 1/\sum_i \exp(\beta(t)e_i)$, after which the first equation gives the connection between $\overline{T}(t)$ and $\beta(t)$. Substituting $\overline{T}(t)$ and $\overline{n}(t)$, expressed in terms of $\beta(t)$, into the second equation of (74) we obtain a differential equation for $\beta(t)$ [25].

We note finally that higher approximations are also constructed in [25] .

Chapter 3

Boundary Value Problems for Singularly Perturbed Equations of Conditionally Stable Type in the Critical Case

In the previous chapters we assumed that the matrix $F_x(\varphi(t,\alpha),t,0)$ evaluated along a family of solutions of the reduced equation had the eigenvalue $\lambda \equiv 0$ of multiplicity k and that its other eigenvalues satisfied the inequality $\operatorname{Re}\lambda < 0$ (Condition III). However, it frequently happens in applied problems (cf. §3) that this matrix also has eigenvalues satisfying $\operatorname{Re}\lambda > 0$ in addition to those with $\lambda \equiv 0$ and $\operatorname{Re}\lambda < 0$. Such cases are naturally called cases of critical conditional stability, and we shall investigate below the associated boundary value problems (as opposed to the initial value problems of Chapters 1 and 2). We examine such problems in this chapter as well as applications of our asymptotic analysis to some concrete systems. In order to do this we will make extensive use of the ideas, methods and results of [13, §14], where we investigated boundary value problems in the "ordinary" conditionally stable cases (that is, $\lambda \equiv 0$ is absent).

The systems of equations considered in this chapter do not have the same general form as those in Chapter 2; instead, we study several important special cases.

§1 Boundary Value Problems for Quasilinear Systems

1. Statement of the Problem. In this subsection we consider the system of equations

$$\mu\frac{dz}{dt} = A(u,t)y + \mu B(u,t) \;,$$

$$\mu\frac{dy}{dt} = z \;, \qquad (0 \le t \le 1) \tag{1}$$

$$\mu\frac{du}{dt} = C(u,t)y + \mu D(u,t) \;,$$

where z and y are scalar functions and u is a k-dimensional vector function. In this case the system is quasilinear because it is linear with respect to z and y. The choice of such a system is motivated in part by its occurrence in the study of applied problems from semiconductor theory.

We prescribe for (1) the following boundary conditions:

$$z(0,\mu) = z^0 \;, \; z(1,\mu) = z^1 \;, \; u(0,\mu) = u^0 \;. \tag{2}$$

I. Suppose that the functions $A(u,t)$, $B(u,t)$, $C(u,t)$ and $D(u,t)$ are sufficiently smooth in some domain $G(u,t)$.

II. Suppose that $A(u,t) > 0$ in $G(u,t)$.

It is clear that the reduced system

$$A(\bar{u},t)\bar{y} = 0 \;, \; \bar{z} = 0 \;, \; C(\bar{u},t)\bar{y} = 0$$

has the family of solutions

$$\bar{z} = 0 \;, \; \bar{y} = 0 \;, \; \bar{u} = \alpha \;,$$

where α is an arbitrary k-dimensional vector. The matrix F_x (evaluated at $z = y = u = \mu = 0$), for

$F = \begin{pmatrix} Ay + \mu B \\ z \\ Cy + \mu D \end{pmatrix}$ and $x = \begin{pmatrix} z \\ y \\ u \end{pmatrix}$, is equal to the block matrix

$$\begin{pmatrix} 0 & A(\alpha,t) & 0 \\ 1 & 0 & 0 \\ 0 & C(\alpha,t) & 0 \end{pmatrix} .$$

It is easy to see that F_x has $\lambda \equiv 0$ as an eigenvalue of multiplicity k as well as two eigenvalues of opposite signs in the domain $G(\alpha,t)$, namely $\lambda_{1,2}(\alpha,t) = \pm\sqrt{A(\alpha,t)}$. Thus we have indeed a critical conditionally stable case. This leads to boundary layers at both ends of the interval $[0,1]$.

2. Construction of the Asymptotic Expansion of the Solution. The asymptotic expansion of the solution of problem (1), (2) will be constructed in the form

$$x(t,\mu) = \bar{x}(t,\mu) + \Pi x(\tau_0,\mu) + Qx(\tau_1,\mu)$$
$$(\tau_0 = t/\mu \ , \ \tau_1 = (t-1)/\mu) \ , \qquad (3)$$

where

$$\bar{x}(t,\mu) = \bar{x}_0(t) + \mu \bar{x}_1(t) + \dots + \mu^n \bar{x}_n(t) + \dots \ ,$$

$$\Pi x(\tau_0,\mu) = \Pi_0 x(\tau_0) + \mu \Pi_1 x(\tau_0) + \dots + \mu^n \Pi_n x(\tau_0) + \dots \ , \qquad (4)$$

$$Qx(\tau,\mu) = Q_0 x(\tau_1) + \mu Q_1 x(\tau_1) + \dots + \mu^n Q_n x(\tau_1) + \dots \ .$$

$\Pi x(\tau_0,\mu)$ and $Qx(\tau_1,\mu)$ represent boundary series at the left and the right ends of the interval $[0,1]$ respectively.

Substituting (3) into (1) and replacing the right-hand side F by the sum $\overline{F} + \Pi F + QF$ (as in [13, §14, Subsection 5]), we obtain a sequence of equations for determining $\overline{x}_i(t)$, $\Pi_i x(\tau_0)$ and $Q_i x(\tau_1)$ $(i = 0,1,\dots)$.

For $\overline{x}_0(t)$ we have the reduced equation

$$A(\overline{u}_0,t)\overline{y}_0 = 0 \ , \ \overline{z}_0 = 0 \ , \ C(\overline{u}_0,t)\overline{y}_0 = 0 \ ,$$

from which we obtain

$$\overline{z}_0 = 0 \ , \ \overline{y}_0 = 0 \ , \ \overline{u}_0 = \alpha(t) \ , \tag{5}$$

where $\alpha(t)$ is an as yet arbitrary k-dimensional vector function.

For $\Pi_0 x(\tau_0)$ there is the system of equations

$$\frac{d\Pi_0 z}{d\tau_0} = A(\alpha(0) + \Pi_0 u,0)\Pi_0 y \ , \quad \frac{d\Pi_0 y}{d\tau_0} = \Pi_0 z \ , \qquad \frac{d\Pi_0 u}{d\tau_0} = C(\alpha(0) + \Pi_0 u,0)\Pi_0 y \ . \tag{6}$$

The initial condition for $\Pi_0 x(\tau_0)$ is obtained after substituting (3) into (2) and has the form

$$\Pi_0 z(0) = z^0 \ , \ \Pi_0 u(0) = u^0 - \alpha(0) \ . \tag{7}$$

As usual, we also require that $\Pi_0 x(\tau_0) \to 0$ as $\tau_0 \to \infty$, that is,

$$\Pi_0 x(\infty) = 0 \ . \tag{8}$$

The as yet arbitrary vector $\alpha(0)$ appears both in the equation (6) and in the initial condition (7). Moreover, the initial value of $\pi_0 y(\tau_0)$ is as yet arbitrary. We will take advantage of this arbitrariness in order to guarantee that condition (8) is satisfied.

To this end let us first describe the stability manifold Ω_0 for system (6) ; it is analogous to the one which figured in our discussions in Chapter 2. From (6) we have that

$$\frac{d\pi_0 u}{d\pi_0 z} = \frac{C(\alpha(0) + \pi_0 u, 0)}{A(\alpha(0) + \pi_0 u, 0)} \quad .$$

Let us denote by

$$\pi_0 u = U_0(\alpha(0), \pi_0 z) \tag{9}$$

the solution of this system such that $\pi_0 u = 0$ for $\pi_0 z = 0$, that is, $U_0(\alpha(0), 0) = 0$. By virtue of Conditions I and II this solution exists and is unique in a certain neighborhood of the point $\pi_0 z = 0$. Substituting it into the first two equations of (6) we obtain the system of equations

$$\begin{aligned} \frac{d\pi_0 z}{d\tau_0} &= A(\alpha(0) + U_0(\alpha(0), \pi_0 z), 0)\pi_0 y \quad , \\ \frac{d\pi_0 y}{d\tau_0} &= \pi_0 z \quad . \end{aligned} \tag{10}$$

The rest point $\pi_0 y = 0$, $\pi_0 z = 0$ of this system is a saddle (that is, conditionally stable), since the roots of the corresponding characteristic equation are clearly equal to $\pm\sqrt{A(\alpha(0), 0)}$, and by virtue of Condition II, are real and have opposite signs. System (10) can be integrated in an elementary fashion by quadratures, and for stability as $\tau_0 \to \infty$ we obtain the equation of the separatrix of the saddle as

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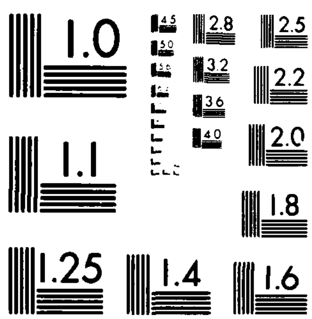

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$$\pi_0 y = -\left(2\int_0^{\pi_0 z} \frac{\xi\, d\xi}{A(\alpha(0) + U_0(\alpha(0),\xi),0)}\right)^{1/2} \operatorname{sgn}(\pi_0 z) \ . \tag{11}$$

The proper choice of the sign in front of the square root (in the present case, minus) was easy to make from an analysis of the phase plane of the variables $\pi_0 z$, $\pi_0 y$. After linearizing the right-hand side of (11) with respect to $\pi_0 z$ we obtain

$$\pi_0 y = -\pi_0 z/\sqrt{A(\alpha(0,0)} \quad .$$

The formulas (9) and (11) give an analytic representation of the one-dimensional manifold Ω_0 having the property that if the initial value $\pi_0 x(0)$ belongs to Ω_0, then $\pi_0 x(\tau_0)$ belongs to Ω_0 for $\tau_0 > 0$. For such a $\pi_0 x(\tau_0)$ the inequality

$$\|\pi_0 x(\tau_0)\| \leq c \exp(-\varkappa\tau_0) \ (\tau_0 \geq 0) \tag{12}$$

is satisfied, which implies the validity of condition (8).

Thus, in order that a solution of system (6) satisfy condition (8) it is necessary to require that the initial value $\pi_0 x(0)$ belong to Ω_0.

III_0. Suppose that the values of $\pi_0 z = z^0$ belong to the domain of definition of the solution (9).

Substituting (7) into (9) we obtain the equation

$$u^0 - \alpha(0) = U_0(\alpha(0), z^0) \quad , \tag{13}$$

which represents a system of k scalar equations in the k unknown components of the vector $\alpha(0)$.

IV. Suppose that equation (13) has a solution $\alpha(0) = \alpha^0$.

If we take $\alpha(0) = \alpha^0$ and define the initial value $\Pi_0 y(0)$ by means of equation (11) (for this we must put $\Pi_0 z = z^0$ and $\alpha(0) = \alpha^0$ in the right-hand side of (11)), then $\Pi_0 x(0)$ belongs to Ω_0 , and consequently, $\Pi_0 x(\tau_0)$ satisfies the inequality (12) and condition (8).

We note that for the actual determination of $\Pi_0 x(\tau_0)$ it is necessary to substitute (11) into the first equation of (10) and to solve the resulting scalar equation for $\Pi_0 z(\tau_0)$ with the initial condition $\Pi_0 z(0) = z^0$. The functions $\Pi_0 u(\tau_0)$ and $\Pi_0 y(\tau_0)$ are determined by formulas (9) and (11) once $\Pi_0 z(\tau_0)$ is found.

Thus $\Pi_0 x(\tau_0)$ is completely determined, while for the as yet unknown function $\bar{u}_0 = \alpha(t)$ we have the initial value α^0 . The function $\alpha(t)$ is determined completely by the following steps.

The equation for $\bar{x}_1(t)$ (that is, for $\bar{z}_1(t)$, $\bar{y}_1(t)$ and $\bar{u}_1(t)$) has the form

$$\frac{d\bar{z}_0}{dt} = A(\bar{u}_0,t)\bar{y}_1 + A_u(\bar{u}_0,t)\bar{y}_0\bar{u}_1 + B(\bar{u}_0,t) \quad ,$$

$$\frac{d\bar{y}_0}{dt} = \bar{z}_1 \quad ,$$

$$\frac{d\bar{u}_0}{dt} = C(\bar{u}_0,t)\bar{y}_1 + C_u(\bar{u}_0,t)\bar{y}_0\bar{u}_1 + D(\bar{u}_0,t) \quad .$$

Hence, by virtue of (5) we have that

$$\bar{z}_1 = 0 \ , \ \bar{y}_1 = -B(\alpha(t),t)/A(\alpha(t),t) \quad , \tag{14}$$

$$\frac{d\alpha}{dt} = -C(\alpha,t)B(\alpha,t)/A(\alpha,t) + D(\alpha,t) \ . \tag{15}$$

Equation (15) is a differential equation for the unknown function $\alpha(t)$.

V. Suppose that equation (15) together with the initial condition $\alpha(0) = \alpha^0$ (see IV) has the solution $\alpha = \alpha(t)$ for $0 \le t \le 1$.

Thus $\bar{x}_0(t)$ is completely determined. Concerning $\bar{x}_1(t)$, the formula (14) defines $\bar{z}_1(t)$ and $\bar{y}_1(t)$, while $\bar{u}_1(t)$ is as yet undetermined, that is, it is possible to set $\bar{u}_1(t) = \beta(t)$, where $\beta(t)$ is an as yet arbitrary k-dimensional vector function.

For $\Pi_1 x(\tau_0)$ we have the system of equations

$$\begin{aligned}
\frac{d\Pi_1 z}{d\tau_0} &= A(\tau_0)\Pi_1 y + A_u(\tau_0)\Pi_0 y(\Pi_1 u + \beta(0)) + \varphi_1(\tau_0) \quad , \\
\frac{d\Pi_1 y}{d\tau_0} &= \Pi_1 z \quad , \\
\frac{d\Pi_1 u}{d\tau_0} &= C(\tau_0)\Pi_1 y + C_u(\tau_0)\Pi_0 y(\Pi_1 u + \beta(0)) + \varphi_2(\tau_0) \quad ,
\end{aligned} \tag{16}$$

where $A(\tau_0) = A(\alpha^0 + \Pi_0 u(\tau_0), 0)$ and analogous meanings are attached to the terms $A_u(\tau_0)$, $C(\tau_0)$ and $C_u(\tau_0)$, while $\varphi_1(\tau_0)$ and $\varphi_2(\tau_0)$ can be expressed in terms of known functions and satisfy the exponential estimates $\|\varphi_i(\tau_0)\| \le c \exp(-\kappa\tau_0)$. The supplementary conditions for $\Pi_1 x(\tau_0)$ have the form

$$\begin{aligned}
&\Pi_1 z(0) = 0 \ , \ \Pi_1 u(0) = -\beta(0) \ , \\
&\Pi_1 x(\infty) = 0 \quad .
\end{aligned} \tag{17}$$

As in the case of $\Pi_0 x(\tau_0)$ we can take advantage of the arbitrariness of $\beta(0)$ and choose it so that condition (17) is satisfied. Let us make the change of variables in the system (16)

$$\pi_1 z = \delta_1 \;,\; \pi_1 y = \delta_2 \;,\; \pi_1 u = \frac{C(\tau_0)}{A(\tau_0)}\,\delta_1 + \delta_3 \quad . \tag{18}$$

It is easy to verify that we obtain the system

$$\begin{aligned}
\frac{d\delta_1}{d\tau_0} &= \frac{A_u(\tau_0)C(\tau_0)}{A(\tau_0)}\,\pi_0 y(\tau_0)\delta_1 + A(\tau_0)\delta_2 \\
&\quad + A_u(\tau_0)\pi_0 y(\tau_0)(\delta_3 + \beta(0)) + \varphi_1(\tau_0) \quad , \\
\frac{d\delta_2}{d\tau_0} &= \delta_1 \;, \\
\frac{d\delta_3}{d\tau_0} &= [C_u(\tau_0) - \frac{C(\tau_0)A_u(\tau_0)}{A(\tau_0)}]\pi_0 y(\tau_0)(\delta_3 + \beta(0)) \\
&\quad + [\varphi_2(\tau_0) - \frac{C(\tau_0)}{A(\tau_0)}\,\varphi_1(\tau_0)] \quad .
\end{aligned} \tag{19}$$

We note that $C(\tau_0)/A(\tau_0)$ plays the role of $H(\tau)$ in Chapter 2, and that the application of this change of variables, as in Chapter 2, leads to a system (19) in which the equation for δ_3 can be separated from those for δ_1 and δ_2 .

The supplementary conditions for δ_1 , δ_2 and δ_3 are

$$\delta_1(0) = 0 \;,\; \delta_3(0) = -\beta(0) \;,\; \delta_i(\infty) = 0 \quad (i = 1,2,3) \quad .$$

The solution of the third equation in (19) and the initial condition $\delta_3(0) = -\beta(0)$ can be written as

$$\delta_3(\tau_0) = -\beta(0) + \int_0^{\tau_0} \Psi(\tau_0)\Psi^{-1}(s)\varphi_3(s)\,ds \quad ,$$

where $\Psi(\tau_0)$ is a fundamental matrix of the corresponding homogeneous system $(\Psi(0) = E_k)$, having the same properties as the function $\Psi(\tau)$ in Lemma 4 of Chapter 2. The function

$$\varphi_3(\tau_0) = \varphi_2(\tau_0) - \frac{C(\tau_0)}{A(\tau_0)}\varphi_1(\tau_0)$$

satisfies an exponential estimate.

The condition $\delta_3(\infty) = 0$ uniquely determines $\beta(0)$:

$$\beta(0) = \Psi(\infty)\int_0^\infty \Psi^{-1}(s)\varphi_3(s)ds \quad .$$

By virtue of the exponential convergence of $\Psi(\tau_0)$ to $\Psi(\infty)$ as $\tau_0 \to \infty$ (Lemma 4, Chapter 2) we obtain for $\delta_3(\tau_0)$ the estimate

$$\|\delta_3(\tau_0)\| \leq c\exp(-\kappa\tau_0) \ (\tau_0 \geq 0) \quad .$$

Having defined $\delta_3(\tau_0)$ we now write the first two equations in (19) as

$$\begin{aligned} \frac{d\delta_1}{d\tau_0} &= \frac{A_u(\tau_0)C(\tau_0)}{A(\tau_0)}\Pi_0 y(\tau_0)\delta_1 + A(\tau_0)\delta_2 + \psi(\tau_0) \ , \\ \frac{d\delta_2}{d\tau_0} &= \delta_1 \ , \end{aligned} \tag{20}$$

where $\psi(\tau_0)$ is an exponentially decreasing function. The homogeneous system corresponding to (20) is the variational system of (10) . Hence, by virtue of Lemma 4.5 in [13], it follows that there exists a unique solution of (20), which satisfies the conditions $\delta_1(0) = 0$, $\delta_i(\infty) = 0$ and which is exponentially decreasing, that is,

$$\|\delta_i(\tau_0)\| \leq c\exp(-\kappa\tau_0) \ (\tau_0 \geq 0 \ ; \ i = 1,2) \quad .$$

Thus $\Pi_1 x(\tau_0)$ is completely determined and satisfies an exponential estimate, while for the as yet unknown function $\beta(t)$ we have found the initial value $\beta(0)$. The complete determination of $\beta(t)$ follows by

steps analogous to those used for the determination of $\alpha(t)$, except that for $\beta(t)$ we obtain a linear differential equation. Thus it is possible to construct the terms $\bar{x}_i(t)$ and $\Pi_i x(\tau_0)$ up to an arbitrary order n .

The determination of the right boundary functions $Q_i x(\tau_i)$ is analogous to the determination of the left boundary functions $\Pi_i x(\tau_0)$. For $Q_0 x(\tau_1)$ we have

$$\frac{dQ_0 z}{d\tau_1} = A(\bar{u}_0(1) + Q_0 u, 1) Q_0 y \ , \ \frac{dQ_0 y}{d\tau_1} = Q_0 z \ , \tag{21}$$

$$\frac{dQ_0 u}{d\tau_1} = C(\bar{u}_0(1) + Q_0 u, 1) Q_0 y$$

and the supplementary conditions

$$Q_0 z(0) = z^1 \ , \quad Q_0 x(-\infty) = 0 \ .$$

A fundamental difference between system (21) and the analogous system (6) for $\Pi_0 x(\tau_0)$ is that $\bar{u}_0(1)$ is a known quantity, while at the same stage $\bar{u}_0(0) = \alpha(0)$ in system (6) is as yet arbitrary. By using this arbitrariness to choose $\alpha(0)$ in a special way (equation (13)), we were able to satisfy conditions (7) and (8). In system (21) there is no such k-dimensional parameter, but the number of supplementary conditions for $Q_0 x(\tau_1)$ is clearly less than the k identities obtained by comparing (7), (8) since $Q_0 u(0)$ is not specified.

From the first two equations in (21) we have

$$\frac{dQ_0 u}{dQ_0 z} = \frac{C(\bar{u}_0(1) + Q_0 u, 1)}{A(\bar{u}_0(1) + Q_0 u, 1)} \ . \tag{22}$$

Let us denote by

$$Q_0 u = U_1(Q_0 z) \tag{23}$$

the solution of equation (22) satisfying the condition $Q_0 u = 0$ for $Q_0 z = 0$. From this we obtain (cf. (11))

$$Q_0 y = \left(2\int_0^{Q_0 z} \frac{\xi\, d\xi}{A(\overline{u}_0(1) + U_1(\xi), 1)}\right)^{1/2} \operatorname{sgn}(Q_0 z) \quad . \tag{24}$$

The formulas (23) and (24) provide an analytic representation of the one-dimensional manifold Ω_1 which is analogous to that for the manifold Ω_0. It is natural then to require that

III_1. The values of $Q_0 z = z^1$ belongs to the domain of definition of the solution (23).

The initial values $Q_0 u(0)$ and $Q_0 y(0)$ are determined by the formulas (23), (24) for $Q_0 z = z^1$, while the solution $Q_0 x(\tau_1)$ of system (21) with these initial conditions belongs to Ω_1 for $\tau_1 \leq 0$ and satisfies the inequality

$$\|Q_0 x(\tau_1)\| \leq c \exp(\varkappa\tau_1) \ (\tau_1 \leq 0) \quad . \tag{25}$$

Consequently it also satisfies $Q_0 x(-\infty) = 0$.

We obtain for $Q_1 x(\tau_1)$ the system

$$\frac{dQ_1 z}{d\tau_1} = A(\tau_1)Q_1 y + A_u(\tau_1)Q_0 y(\tau_1)Q_1 u + \psi_1(\tau_1) \quad ,$$

$$\frac{dQ_1 y}{d\tau_1} = Q_1 z \quad , \tag{26}$$

$$\frac{dQ_1 u}{d\tau_1} = C(\tau_1)Q_1 y + C_u(\tau_1)Q_0 y(\tau_1)Q_1 u + \psi_2(\tau_1) \quad ,$$

and the supplementary conditions

$$Q_1 z(0) = -\bar{z}_1(1) = 0 \ , \quad Q_1 x(-\infty) = 0 \ , \tag{27}$$

where $A(\tau_1) = A(\bar{u}_0(1) + Q_0 u(\tau_1), 1)$ and analogous meanings are ascribed to $A_u(\tau_1)$, $C(\tau_1)$ and $C_u(\tau_1)$. The functions $\psi_1(\tau_1)$ and $\psi_2(\tau_2)$ are known and satisfy an exponential estimate of the type (25).

By means of a change of variables like (18) it is a simple matter to prove that $Q_1 x(\tau_1)$ exists and satisfies an exponential estimate like (25).

Succeeding terms $Q_i x(\tau_1)$ follow in an analogous manner.

3. An Estimate of the Remainder Term. Let us introduce in the space of the variables (x,t) a curve L composed of the three pieces:

$$L_1 = \{(x,t)\colon \ x = \bar{x}_0(0) + \Pi_0 x(\tau_0)(\tau_0 \geq 0); \ t = 0\} \ ,$$

$$L_2 = \{(x,t)\colon \ x = \bar{x}_0(t); \ 0 \leq t \leq 1\} \ ,$$

$$L_3 = \{(x,t)\colon \ x = \bar{x}_0(1) + Q_0 x(\tau_1)(\tau_1 \leq 0); \ t = 1\} \ .$$

We denote by ℓ the projection of this curve onto the space of the variables (u,t) . It is possible to take for the domain $G(u,t)$ occurring in Condition I an arbitrary δ-tube of the curve ℓ . More precisely,

I. Suppose that the functions $A(u,t)$ and $C(u,t)$ $(B(u,t)$ and $D(u,t))$ have continuous partial derivatives with respect to each argument up to order $(n+2)$ $((n+1))$ inclusive in some δ-tube of the curve ℓ .

Having defined the terms of the series (4) up to order $(n+1)$ inclusive, let us now denote by $X_k(t,\mu)$ the k-th partial sum of the series (3), that is,

$$X_k(t,\mu) = \sum_{i=0}^{k} \mu^i(\bar{x}_i(t) + \pi_i x(\tau_0) + Q_i x(\tau_1)) \quad . \tag{28}$$

Theorem 3. Under Conditions I-V there exist positive constants μ_0 and c such that for $0 < \mu \leq \mu_0$ there exists a unique solution $x(t,\mu)$ of the boundary value problem (1), (2) lying in a $c\mu$-tube of the curve L and satisfying the inequality

$$\|x(t,\mu) - X_n(t,\mu)\| \leq c\mu^{n+1} (0 \leq t \leq 1) \quad . \tag{29}$$

Proof. Let us set $\zeta = z - Z_{n+1}$, $\eta = y - Y_{n+1}$ and $w = u - U_{n+1}$, where z, y and u is the unknown solution of the problem (1), (2), and Z_{n+1}, Y_{n+1} and U_{n+1} are the partial sums determined by (28). By substituting these into (1), (2) we obtain for ζ , η and w the boundary value problem

$$\mu\frac{d\zeta}{dt} = A(U_0,t)\eta + A_u(U_0,t)Y_0(t,\mu)w + G_1(\eta,w,t,\mu) \quad ,$$

$$\mu\frac{d\eta}{dt} = \zeta + O(\mu^{n+2}) \quad , \tag{30}$$

$$\mu\frac{dw}{dt} = C(U_0,t)\eta + C_u(U_0,t)Y_0(t,\mu)w + G_2(\eta,w,t,\mu) \quad ,$$

$\zeta(0,\mu)$, $\zeta(1,\mu)$ and $w(0,\mu)$ are known and of order $O(\mu^{n+2})$.

In the equations of (30) we have isolated the linear terms whose coefficients are evaluated at the zeroth approximation. The functions

$$G_1(\eta,w,t,\mu) = A(U_{n+1} + w,t)(Y_{n+1} + \eta)$$

$$+ \mu B(U_{n+1} + w,t) - \mu \frac{dU_{n+1}}{dt}$$

$$- A(U_0,t)\eta - A_u(U_0,t)Y_0(t,\mu)w$$

and $G_2(\eta,w,t,\mu)$, which is defined analogously, satisfy the following two important properties:

1. $G_i(0,0,t,\mu) = \Theta(\mu^{n+2})$;

2. $G_i(\eta,w,t,\mu)$ is a contraction operator with contraction coefficient of order $\Theta(\mu)$ for η and w of order $\Theta(\mu)$.

It is necessary to transform the $G_i(\eta,w,t,\mu)$ into a different form for the subsequent analysis. We begin with the identity

$$A(U_{n+1} + w,t) = A(U_{n+1},t) + A_u(U_{n+1},t)w$$

$$+ [A(U_{n+1} + w,t) - A(U_{n+1},t) - A_u(U_{n+1},t)w]$$

$$\equiv A(U_{n+1},t) + A_u(U_{n+1},t)w + q_1(w,t,\mu) .$$

Here the function $q_1(w,t,\mu)$ is clearly a contraction operator with contraction coefficient of order $\Theta(\mu)$ for w of order $\Theta(\mu)$; moreover , $q_1(0,t,\mu) = 0$. By expressing $C(U_{n+1} + w,t)$ in an analogous form (corresponding to $q_1(w,t,\mu)$ there is a contraction operator which we denote

by $q_2(w,t,\mu))$ and doing the same for $B(U_{n+1}+w,t)$ and $D(U_{n+1}+w,t)$, the functions $G_i(\eta,w,t,\mu)$ can be reduced to the form

$$G_i(\eta,w,t,\mu) = \mu\, a_i(t,\mu)\eta + \mu\, b_i(t,\mu)w + c_i(t,\mu)\eta w$$

$$+ q_i(w,t,\mu)Y_0(t,\mu) + Q_i(\eta,w,t,\mu) \quad ,$$

where a_i, b_i and c_i are certain bounded functions or matrices (Here and below for ease of writing we will denote a function or matrix by one and the same symbol ω since only the boundedness of this quantity is important to us.), and $Q_i(\eta,w,t,\mu)$ is a contraction operator with contraction coefficient of order $\Theta(\mu^2)$ for η and w of order $\Theta(\mu)$. In addition, $Q_i(0,0,t,\mu) = \Theta(\mu^{n+2})$.

Let us now replace $w(t,\mu)$ in the system (30) by the function $\xi(t,\mu)$, where $w = \xi + (C(U_0,t)/A(U_0,t))\zeta$. An elementary calculation shows that this system of equations assumes the form

$$\mu\frac{d\zeta}{dt} = \frac{A_u(U_0,t)C(U_0,t)}{A(U_0,t)}\, Y_0(t,\mu)\zeta + A(U_0,t)\eta$$

$$+ [\omega Y_0(t,\mu)\xi + G_1(\eta,\xi+\frac{C(U_0,t)}{A(U_0,t)}\,\zeta,t,u)] \quad , \tag{31}$$

$$\mu\frac{d\eta}{dt} = \zeta + \Theta(\mu^{n+2}) \quad ,$$

$$\mu\frac{d\xi}{dt} = h(t,\mu)\xi + [G(\xi,\eta,\zeta,t,\mu) + q(\xi,\zeta,t,\mu)Y_0(t,\mu)$$

$$+ Q(\xi,\eta,\zeta,t,\mu)] \quad ,$$

where

$$h(t,\mu) = \Theta(\mu + \exp(-\varkappa t/\mu) + \exp(-\varkappa(1-t)/\mu)) \quad , \tag{32}$$

$$G(\xi,\eta,\zeta,t,\mu) = \mu\omega\eta + \mu\omega\zeta + \mu\omega\xi\eta + \mu\omega\eta\zeta \quad ,$$

$$q(\xi,\zeta,t,\mu) = -\frac{C(U_0,t)}{A(U_0,t)} q_1(\xi + \frac{C(U_0,t)}{A(U_0,t)}\zeta,t,\mu) + q_2(\xi + \frac{C(U_0,t)}{A(U_0,t)}\zeta,t,\mu) \quad ,$$

$$Q(\xi,\eta,\zeta,t,\mu) = -\frac{C(U_0,t)}{A(U_0,t)} Q_1(\eta,\xi + \frac{C(U_0,t)}{A(U_0,t)}\zeta,t,\mu) + Q_2(\eta,\xi + \frac{C(U_0,t)}{A(U_0,t)}\zeta,t,\mu) \quad .$$

The operator $q(\xi,\zeta,t,\mu)$ is a contraction with contraction coefficient of order $\Theta(\mu)$ for ξ and ζ of order $\Theta(\mu)$ satisfying $q(0,0,t,\mu) = 0$, while the operator Q is also a contraction with contraction coefficient of order $\Theta(\mu^2)$ for ξ, η and ζ of order $\Theta(\mu)$ satisfying $Q(0,0,0,t,\mu) = \Theta(\mu^{n+2})$.

We will consider the terms contained in the square brackets of the equations in (31) as nonhomogeneous terms by passing from system (31) to an equivalent system of integral equations. Let us denote by $\Gamma(t,s,\mu)$ the Green's matrix for the boundary value problem consisting of the first two equations in (31) together with the boundary conditions $\zeta(0,\mu) = \zeta(1,\mu) = 0$. It is possible to prove as in [8] that the Green's matrix exists and satisfies the estimate

$$\Gamma(t,s,\mu) = \Theta(\exp(-\varkappa|t-s|/\mu)) \quad .$$

The solution of the corresponding homogeneous system and the boundary conditions $\zeta(0,\mu) = \Theta(\mu^{n+2})$, $\zeta(1,\mu) = \Theta(\mu^{n+2})$ has the same order of smallness as the boundary values. In place of the first two equations in (31) we have therefore the integral equation

$$\begin{pmatrix} \zeta(t,\mu) \\ \eta(t,\mu) \end{pmatrix} = \Theta(\mu^{n+2}) + \int_0^1 \mu^{-1} T(t,s,\mu) \begin{pmatrix} \omega Y_0(s,\mu)\xi(s,\mu) + G_1 \\ \Theta(\mu^{n+2}) \end{pmatrix} ds$$

$$\equiv \begin{pmatrix} S_1(\xi,\eta,\zeta,t,\mu) \\ S_2(\xi,\eta,\zeta,t,\mu) \end{pmatrix} . \tag{33}$$

Let us denote by $H(t,s,\mu)$ the fundamental matrix of the homogeneous system $\mu\, d\xi/dt = h(t,\mu)\xi$. By virtue of (32) $H(t,s,\mu)$ is bounded. The initial condition for $\xi(t,\mu)$ is clearly of the same type as that for $w(t,\mu)$, that is, $\xi(0,\mu) = \Theta(\mu^{n+2})$. Therefore the last equation in (31) can be written as the integral equation

$$\xi(t,\mu) = \Theta(\mu^{n+2}) + \mu^{-1} \int_0^t H(t,s,\mu)[G(\xi,\eta,\zeta,s,\mu) + q(\xi,\zeta,s,\mu)Y_0(s,\mu) + Q(\xi,\eta,\zeta,s,\mu)]ds . \tag{34}$$

The operator

$$R_1(\xi,\eta,\zeta,t,\mu) \equiv \mu^{-1} \int_0^t H(t,s,\mu)Q\, ds$$

by virtue of the properties of Q is a contraction with contraction coefficient of order $\Theta(\mu)$ for ξ, η and ζ of order $\Theta(\mu)$; moreover, $R_1(0,0,0,t,\mu) = \Theta(\mu^{n+1})$. Since

$$Y_0(t,\mu) = \Theta(\exp(-\varkappa t/\mu) + \exp(-\varkappa(1-t)/\mu)) \quad ,$$

and therefore

$$\int_0^t H(t,s,\mu)Y_0(s,\mu)ds = \int_0^t \Theta(\exp(-\varkappa s/\mu) + \exp(-\varkappa(1-s)/\mu))ds$$

$$= \Theta(\mu) \quad ,$$

the operator

$$R_2(\xi,\eta,\zeta,t,\mu) \equiv \mu^{-1}\int_0^t H(t,s,\mu)q(\xi,\zeta,s,\mu)Y_0(s,\mu)ds$$

has the same properties as $R_1(\xi,\eta,\zeta,t,\mu)$. Let us now set $R(\xi,\eta,\zeta,t,\mu) = R_1 + R_2 + \Theta(\mu^{n+2})$ and substitute into the expression for G the values of ζ and η from formula (33) . Then in place of (34) we obtain the equation

$$\xi(t,\mu) = \int_0^t H(t,s,\mu)(\omega S_1 + \omega S_2 + \frac{\omega}{\mu}\xi S_2 + \frac{\omega}{\mu}S_1 S_2)ds$$

$$+ R(\xi,\eta,\zeta,t,\mu) \quad . \tag{35}$$

By taking account of the estimate for the Green's function, namely

$$\Gamma(t,s,\mu) = \Theta(\exp(-\varkappa|t-s|/\mu) \quad ,$$

the estimate for $Y_0(t,\mu)$ and the fact that

$$\int_0^t\int_0^1 \mu^{-1}\exp(-\varkappa|s-p|\mu)[\exp(-\varkappa p/\mu) + \exp(-\varkappa(1-p)/\mu)]dp\,ds$$

$$= \Theta(\mu) \quad ,$$

it is easy to show that the first term in the right-hand side of (35) is a contraction operator of the same type as the second term $R(\xi,\eta,\zeta,t,\mu)$. Thus, equation (35) can be written as

$$\xi(t,\mu) = T_1(\xi,\eta,\zeta,t,\mu) \quad , \tag{36}$$

where the operator $T_1(\xi,\eta,\zeta,t,\mu)$ is a contraction with contraction coefficient of order $\Theta(\mu)$ for ξ,η and ζ of order $\Theta(\mu)$; moreover, $T_1(0,0,0,t,\mu) = \Theta(\mu^{n+1})$.

Substituting (36) into (33) we obtain the equations

$$\begin{aligned} \zeta(t,\mu) &= S_1(T_1,\eta,\zeta,t,\mu) \equiv T_2(\xi,\eta,\zeta,t,\mu) \quad , \\ \eta(t,\mu) &= S_2(T_1,\eta,\zeta,t,\mu) \equiv T_3(\xi,\eta,\zeta,t,\mu) \quad , \end{aligned} \tag{37}$$

in which the operators T_2 and T_3 are similar to T_1.

We now apply to the system (36), (37) the method of successive approximations as in [13]. It is possible to prove that for sufficiently small μ a unique solution exists in a certain $c\mu$- tube of the curve $\xi = \eta = \zeta = 0$, and satisfies the estimates $\xi = \Theta(\mu^{n+1})$, $\eta = \Theta(\mu^{n+1})$, $\zeta = \Theta(\mu^{n+1})$. Hence, it follows also that $w = \Theta(\mu^{n+1})$.

Thus $z - Z_{n+1}$, $y - Y_{n+1}$ and $u - U_{n+1}$ are all of order $\Theta(\mu^{n+1})$, and since $X_{n+1} - X_n = \Theta(\mu^{n+1})$ the inequality (29) is established. This completes the proof of the theorem.

§2 Other Boundary Value Problems

1. Boundary Value Problems of a More General Type. In §1 we considered a problem with the boundary conditions (2) . Using the results for this problem it is possible to consider more general boundary conditions. The corresponding constructions are analogous to those which were performed in detail in [13, §13] for the case $\operatorname{Re}\lambda_i < 0$ and in [23] for the conditionally stable case. Therefore we confine ourselves to a brief description of the constructive scheme.

Suppose that the boundary conditions for system (1) are of the form

$$R(x(0,\mu),\ x(1,\mu))$$

$$\equiv R(z(0,\mu),\ y(0,\mu),\ u(0,\mu),\ z(1,\mu),\ y(1,\mu),\ u(1,\mu)) = 0\ , \qquad (38)$$

in which the dimension of the vector R is equal to $k+2$, the dimension of x . We consider as an auxiliary problem the boundary value problem (1), (2) with as yet arbitrary values of z^0, z^1 and u^0 . We propose to select z^0, z^1 and u^0 so that the solution of the problem (1), (2) satisfies the condition in (38) . This device was used in [13, §13] .

Let us seek z^0 , z^1 and u^0 in the form of power series in μ ; for example,

$$z^0 = z^0_0 + \mu z^0_1 + \mu^2 z^0_2 + \dots \quad .$$

Under Conditions I - V we can construct an asymptotic expansion of the solution and substitute it into equation (38) . By further decomposing $R(x(0,\mu),\ x(1,\mu))$ into a power series in μ we obtain equations for the terms in the series for z^0, z^1 and u^0 . Thus, in the zeroth

approximation, we have the equation (for simplicity of notation we omit the lower index 0, that is, we write z^0 in place of z_0^0 for example)

$$R(z^0, \pi_0 y(0), u^0, z^1, Q_0 y(0), \bar{u}_0(1) + Q_0 u(0)) = 0 \quad . \tag{39}$$

We note that $\pi_0 y(0)$ as defined by formula (11) for $\pi_0 z = z^0$ is a function of z^0 and $\alpha(0)$. In turn $\alpha(0)$ is defined by equation (13) as a function of z^0 and u^0. Thus $\pi_0 y(0)$ is a known function of z^0 and u^0. Similarly, $\bar{u}_0(1)$ is a known function of z^0 and u^0, while $Q_0 u(0)$ and $Q_0 y(0)$ are defined by the formulas (23) and (24) for $Q_0 z = z^1$. Hence it follows that $Q_0 u(0)$ and $Q_0 y(0)$ are known functions of z^1, z^0 and u^0. This dependence on z^0 and u^0 results from the fact that $\bar{u}_0(1)$ enters into equations (23) and (24). Thus the equation (39) is a (k+2)-dimensional vector equation in the (k+2) unknowns: z^0, z^1 and k components of the vector u^0. If the equation (39) has a solution $z^0 = z_0^0$, $z^1 = z_0^1$ and $u^0 = u_0^0$ and if the corresponding functional determinant $D(R)/D(z^0, z^1, u^0)$ is not zero at the point (z_0^0, z_0^1, u_0^0), then each of the succeeding equations can be solved for z_i^0, z_i^1, u_i^0 $(i = 1,2,\ldots)$. Moreover, for sufficiently small values of μ there exists in a certain δ-tube of the point (z_0^0, z_0^1, u_0^0) a unique point $(z^0(\mu), z^1(\mu), u^0(\mu))$ such that the solution of the equation (1) and the boundary condition

$$z(0,\mu) = z^0(\mu) \ , \ z(1,\mu) = z^1(\mu) \ , \ u(0,\mu) = u^0(\mu) \tag{40}$$

satisfies the boundary condition (38). The formally constructed series

$$z_0^0 + \mu z_1^0 + \ldots, \; z_0^1 + \mu z_1^1 + \ldots, \; u_0^0 + \mu u_1^0 + \ldots$$

are asymptotic series for $z^0(\mu)$, $z^1(\mu)$ and $u^0(\mu)$, while the asymptotic expansion of the solution of the problem (1), (40) serves as an asymptotic expansion for the basic problem (1), (38). The proofs of these assertions can be given without difficulty by using the methods in [13, §13].

2. <u>A Class of Boundary Value Problems Reducible to a Type Already Considered</u>. Suppose that a singularly perturbed system has the form

$$\mu^2 \frac{dy}{dt} = F(u,t) \;, \quad \frac{du}{dt} = C(u,t)y + D(u,t) \;, \tag{41}$$

where y is a scalar and u a vector, and suppose that a certain boundary value problem is posed for the system (41). For definiteness we will consider the following boundary conditions

$$u(0,\mu) = u^0 \;, \; y(1,\mu) = y^1 \;. \tag{42}$$

(It is of course possible to consider other types.) The peculiar thing about the system (41) is that the function F does not depend on y, and therefore the usual algorithm for the construction of the asymptotic solution of a singularly perturbed problem is inapplicable here. This follows because for $\mu = 0$ the equation $F(u,t) = 0$ cannot be solved for y. One way of circumventing this difficulty is the following. We differentiate the first equation in (41) and use the second equation to obtain

$$\mu^2 \frac{d^2y}{dt^2} = F_u(u,t)[C(u,t)y + D(u,t)] + F_t(u,t)$$

$$\equiv A(u,t)y + B(u,t) \quad .$$

If we now introduce the new variable $z = \mu\frac{dy}{dt}$, then we are led to the system

$$\mu\frac{dz}{dt} = A(u,t)y + B(u,t), \ \mu\frac{dy}{dt} = z \ ,$$

$$\frac{du}{dt} = C(u,t)y + D(u,t) \quad . \tag{43}$$

It is necessary to prescribe for the system (43) another condition besides the boundary conditions (42), since as a result of differentiation the order of the system has increased by one. This condition is obtained from the first equation in (41) by setting $t = 0$:

$$z(0,\mu) = \frac{F(u^0,0)}{\mu} \quad .$$

Thus the boundary conditions for system (43) have a singularity as $\mu \to 0$. However, it is possible to remove this singularity (cf. Chapter 2, §4, Subsection 1) by introducing the new variables $\tilde{z} = \mu z$, $\tilde{y} = \mu y$. For these new variables we have

$$\mu\frac{d\tilde{z}}{dt} = A(u,t)\tilde{y} + \mu B(u,t), \ \mu\frac{d\tilde{y}}{dt} = \tilde{z} \ , \tag{44}$$

$$\mu\frac{du}{dt} = C(u,t)\tilde{y} + \mu D(u,t) \quad ,$$

which coincides with (1) except for notation. The boundary conditions for the new variables are now regular in μ , and so we can use the method described in Subsection 1 for the construction of the asymptotic expansion of the solution.

We note that the passage from system (41) to system (44) is not the only one which allows the construction of the asymptotic expansion. It is possible to apply to the original problem (41), (42) a certain modified algorithm for the construction of the expansion in the form of a regular part and a boundary part. For simplicity let us consider the case when both y and u are scalar functions. We seek a solution of the problem (41), (42) in the following form

$$y(t,\mu) = \bar{y}_0(t) + \mu\bar{y}_1(t) + \dots + \mu^{-1}\Pi_{-1}y(\tau_0) + \Pi_0 y(\tau_0) + \dots + Q_0 y(\tau_1) + \mu Q_1 y(\tau_1) + \dots ,$$

$$u(t,\mu) = \bar{u}_0(t) + \mu\bar{u}_1(t) + \dots + \Pi_0 u(\tau_0) + \mu\Pi_1 u(\tau_0) + \dots + Q_0 u(\tau_1) + \mu Q_1 u(\tau_1) + \dots .$$

Then for $\bar{y}_0(t)$ and $\bar{u}_0(t)$ we obtain the system

$$0 = F(\bar{u}_0,t), \quad \frac{d\bar{u}_0}{dt} = C(\bar{u}_0,t)\bar{y}_0 + D(\bar{u}_0,t) . \tag{45}$$

Suppose that $\varphi(t) = \bar{u}_0(t)$ is a certain root of the first equation in (45). Substituting it into the second equation gives $\bar{y}_0$:

$$\bar{y}_0 = \frac{\varphi'(t) - D(\varphi(t),t)}{C(\varphi(t),t)} .$$

Thus the leading term of the regular part of the asymptotic expansion is determined in a rather unusual way. For $\Pi_{-1}y(\tau_0)$ and $\Pi_0 u(\tau_0)$ we obtain the system

$$\frac{d\Pi_1 y}{d\tau_0} = F(\varphi(0) + \Pi_0 u, 0) \;,\; \frac{d\Pi_0 u}{d\tau_0} = C(\varphi(0) + \Pi_0 u, 0)\Pi_{-1} y \qquad (46)$$

together with the supplementary conditions

$$\Pi_0 u(0) = u^0 - \varphi(0) \;,\; \Pi_{-1} y(\infty) = 0 \;,\; \Pi_0 u(\infty) = 0 \;. \qquad (47)$$

If $F_u(\varphi(0),0)\, C(\varphi(0),0) > 0$ (In order to obtain an asymptotic expansion of the desired form it is natural to require that $F_u(\varphi(t),t)C(\varphi(t),t) > 0$; such a condition appears in the first approach considered for the system (41)), then the rest point $\Pi_{-1} y = \Pi_0 u = 0$ of system (46) will be conditionally stable. The stability separatrix for $\tau_0 \to \infty$ is described by the equation

$$\Pi_{-1} y = -\left(2 \int_0^{\Pi_0 u} \frac{F(\varphi(0) + \xi, 0)}{C(\varphi(0) + \xi, 0)}\, d\xi\right)^{1/2} \operatorname{sgn} C(\varphi(0),0)\, \operatorname{sgn} \Pi_0 u \;.$$

By substituting $\Pi_0 u(0) = u^0 - \varphi(0)$ into the right-hand side we obtain the initial value $\Pi_{-1} y(0)$. The solution of system (46) with these initial values satisfies each of the conditions in (47) and an exponential estimate.

The function $Q_0 u(\tau_1)$ is found to be identically zero (This is quite natural since the function u is not given at the point $t = 1$.), while for $Q_0 y(\tau_1)$ and $Q_1 u(\tau_1)$ we obtain the linear system

$$\frac{dQ_0 y}{d\tau_1} = F_u(\varphi(1),1)Q_1 u \;,\; \frac{dQ_1 u}{d\tau_1} = C(\varphi(1),1)Q_0 y \qquad (48)$$

along with the supplementary conditions

$$Q_0 y(0) = y^1 - \bar{y}_0(1) \;,\; Q_0 y(-\infty) = 0 \;,\; Q_1 u(-\infty) = 0 \;. \qquad (49)$$

Since $F_u(\varphi(1),1)\ C(\varphi(1),1) > 0$ the characteristic equation for system (48) has roots $\lambda_{1,2} = \pm\sqrt{F_u C}$ of opposite signs, and the solution satisfying the conditions in (49) has the form

$$Q_0 y(\tau_1) = (y^1 - \bar{y}_0(1))\exp(\lambda\tau_1) \ ,$$

$$Q_1 u(\tau_1) = \frac{(y^1 - \bar{y}_0(1))\lambda}{F_u(\varphi(1),1)} \exp(\lambda\tau_1), \text{ for } \lambda = \sqrt{F_u C} \ .$$

One can now construct the succeeding terms of the asymptotic expansion in a similar manner.

3. Boundary Value Problems for a Weakly Nonlinear Equation. In Chapter 1 we considered the initial value problem for the weakly nonlinear equation

$$\mu\frac{dx}{dt} = A(t)x + \mu f(x,t,\mu) \quad (0 \le t \le T)$$

under the assumption that the matrix $A(t)$ had the eigenvalue $\lambda(t) \equiv 0$ of multiplicity k and that the remaining eigenvalues $\lambda_i(t)$ satisfied $\operatorname{Re}\lambda_i(t) < 0$. If now in addition to the zero eigenvalue of multiplicity k the matrix $A(t)$ has m_1 eigenvalues $\lambda_i(t)$ such that $\operatorname{Re}\lambda_i(t) < 0$ and m_2 eigenvalues $\lambda_i(t)$ such that $\operatorname{Re}\lambda_i(t) > 0$ (with $k + m_1 + m_2 = m$) , then we would like to obtain the same qualitative results as in Chapter 1. In other words, we want to determine which solution of the reduced equation $A(t)\bar{x} = 0$ is the limit as $\mu \to 0$ of the solution $x(t,\mu)$. To achieve this it is necessary to consider in place of an initial value problem a boundary value problem in which at least m_1 components of

x are prescribed at $t = 0$ and at least m_2 components are prescribed at $t = T$. The asymptotic expansion of the solution will have the form (3), but the terms of the expansion are determined using certain modifications in the construction procedure which are analogous to those employed in Chapter 1. The details are given in [16], where a similar boundary value problem for a weakly nonlinear system of difference equations is considered.

§3 Applications

1. A Problem from the Theory of Transistors. We first make some explanatory remarks of a physical nature. Consider a contact $(w = 0)$ between two semiconductors of different types, leading to a one-dimensional problem. To the left of the contact $(-\ell \leq w \leq 0)$ we place a semiconductor of p-type, while to the right $(0 \leq w \leq \ell)$ a semiconductor of n-type, that is, a (p-n) junction. Such a semiconductor scheme can be described by a system of equations, consisting of Poisson's equation

$$\frac{\partial E}{\partial w} = \frac{q}{\epsilon}(p - n + N_D^+ - N_A^-) \tag{50}$$

(Here E is the polar electric voltage, p, n, N_D^+ and N_A^- are the respective concentrations of holes, electrons, donors and acceptors, q is the electron charge, and ϵ is the dielectric permeability.) and the equations for the holes (i_p) and the electron current (i_n)

$$\begin{aligned} i_p &= q\mu_p EP - qD_p \frac{\partial p}{\partial w} , \\ i_n &= q\mu_n E_n + qD_n \frac{\partial n}{\partial w} . \end{aligned} \tag{51}$$

(Here μ_p, μ_n are the mobilities of holes and electrons, respectively, and D_p, D_r are the diffusion coefficients of holes and electrons.)

It is known that $\mu_p/D_p = \mu_n/D_n = q/kT$, where k is Boltzmann's constant and T is the temperature. We will assume that the problem is stationary in time and that there are no externally generated sources. Then from the continuity equation it follows that i_p and i_n are constant.

We will consider the following special case, namely, to the left of the contact $N_A^- = N, N_D^+ = 0$, while to the right $N_A^- = 0$, $N_D^+ = N$. Let us now introduce the dimensionless variables

$$t = w/\ell,\ \tilde{y} = E\ell q/kT,\ v_1 = p/N,\ v_2 = n/N \quad ,$$

$$i_p\ell/N_q D_p = c_1,\ i_n\ell/N_q D_n = c_2,\ \mu^2 = \epsilon kT/q^2 N\ell^2 \quad .$$

Then the system (50) - (51) can be written in dimensionless form as

$$\begin{aligned} &\mu^2 \frac{d\tilde{y}}{dt} = v_1 - v_2 + N(t) \ , \\ &\frac{dv_1}{dt} = v_1\tilde{y} - c_1,\ \frac{dv_2}{dt} = -v_2\tilde{y} + c_2 \ , \end{aligned} \tag{52}$$

where $N(t) = \begin{cases} -1, & -1 \leq t < 0 \\ 1, & 0 < t \leq 1 \end{cases}$, and μ is a small quantity of order 10^{-2}.

It is possible to consider various kinds of boundary conditions for the system (52). We restrict ourselves here to one of the simplest, known as the symmetric case, in which $c_1 = c_2 = c$ is a given constant,

and so the boundary conditions are given separately for the intervals $[-1,0]$ and $[0,1]$; namely,

$$v_1(-1) = 1 , \ v_2(-1) = 0 , \ v_1(0) = v_2(0) , \tag{53}$$

and

$$v_1(0) = v_2(0) , \ v_1(1) = 0 , \ v_2(1) = 1 . \tag{54}$$

In the present case the problem (52), (53) reduces to the problem (52), (54) under the changes of variables $t \to -t$, $v_1 \to v_2$ and $v_2 \to v_1$. Therefore it suffices to consider only one of these problems. We will consider the system

$$\begin{aligned} \mu^2 \frac{d\tilde{y}}{dt} &= v_1 - v_2 + 1 , \\ \frac{dv_1}{dt} = v_1\tilde{y} - c , \ \frac{dv_2}{dt} &= -v_2\tilde{y} + c , \end{aligned} \quad (0 \le t \le 1) \tag{55}$$

along with the conditions in (54) .

By introducing the new variables $u_1 = v_1 + v_2$, $u_2 = v_1 - v_2$ we obtain the system

$$\begin{aligned} \mu^2 \frac{d\tilde{y}}{dt} &= u_2 + 1 , \\ \frac{du_1}{dt} = u_2\tilde{y} , \ \frac{du_2}{dt} &= u_1\tilde{y} - 2c , \end{aligned} \tag{56}$$

which is clearly of the same type as (41) since the right-hand side of the first equation does not contain $\tilde{y}$. Proceeding as in Subsection 2 of §2 , that is, differentiating the first equation and introducing the new variables

$$y = \mu\tilde{y} , \ z = \mu\frac{dy}{dt} ,$$

we obtain the system

$$\mu \frac{dz}{dt} = u_1 y - 2\mu c \ , \ \mu \frac{dy}{dt} = z \ ,$$
$$\mu \frac{du_1}{dt} = u_2 y \ , \ \mu \frac{du_2}{dt} = u_1 y - 2\mu c \ . \tag{57}$$

This system is of the type studied in §1 , where the dimension of the vector u is now two. The boundary conditions (54) in the new variables have the form

$$u_2(0) = 0 \ , \ u_1(1) = 1 \ , \ u_2(1) = -1 \ . \tag{58}$$

In addition, it is necessary to supply a further condition, which is obtained from the first equation (56) in the two forms

$$z(0) = 1 \ , \tag{59}$$

and

$$z(1) = 0 \ . \tag{60}$$

It is easy to see that of the five conditions (58), (59) and (60) we need only consider the four

$$z(0) = 1 \ , \ z(1) = 0 \ , \ u_1(1) = 1 \ , \ u_2(1) = -1 \ , \tag{61}$$

since the condition $u_2(0) = 0$ is automatically satisfied. Indeed, from the first and last equations in (57) it follows that $dz/dt = du_2/dt$; whence, noting the boundary conditions $z(1) = 0$, $u_2(1) = -1$, we obtain that $z = u_2 + 1$. Hence, by virtue of the condition $z(0) = 1$ it follows that $u_2(0) = 0$.

Thus we have a boundary value problem (57), (61) which is of the same type as the problem (1), (2) in §1 , the only exception being that the boundary condition for u is given at $t = 1$ rather than at $t = 0$.

Let us now construct the asymptotic expansion of the solution as in §1 . We have first that

$$\bar{z}_0 = 0 , \bar{y}_0 = 0 , \bar{u}_1 = \alpha_1(t) , \bar{u}_2 = \alpha_2(t) ,$$

where α_1 and α_2 are as yet arbitrary functions. Since u is prescribed at $t = 1$ we consider first the system of equations for $Q_0x(\tau_1)$ $(\tau_1 = (t-1)/\mu)$ (Note that the roles of Πx and Qx in the present problem are interchanged relative to §1 .)

$$\frac{dQ_0z}{d\tau_1} = (\alpha_1(1) + Q_0u_1)Q_0y , \quad \frac{dQ_0y}{d\tau_1} = Q_0z ,$$

$$\frac{dQ_0u_1}{d\tau_1} = (\alpha_2(1) + Q_0u_2)Q_0y , \quad \frac{dQ_0u_2}{d\tau_1} = (\alpha_1(1) + Q_0u_1)Q_0y . \tag{62}$$

The supplementary conditions for Q_0x are

$$Q_0z(0) = 0 , \; Q_0u_1(0) = 1 - \alpha_1(1) , \; Q_0u_2(0) = -1 - \alpha_2(1) ,$$

$$Q_0x(-\infty) = 0 . \tag{63}$$

Let us construct the manifold Ω_1 for the system (62). We obtain from (62) the equations

$$\frac{dQ_0u_1}{dQ_0z} = \frac{\alpha_2(1) + Q_0u_2}{\alpha_1(1) + Q_0u_1} , \quad \frac{dQ_0u_2}{dQ_0z} = 1 .$$

The solution of this system and the initial condition $Q_0 u = 0$ for $Q_0 z = 0$ is

$$Q_0 u_1 = -\alpha_1(1) + (\operatorname{sgn}\alpha_1(1))\,(\alpha_1^2(1) + 2\alpha_2(1) Q_0 z + Q_0^2 z)^{1/2} , \tag{64}$$

$$Q_0 u_2 = Q_0 z \quad .$$

Equation (64) is analogous to equation (9) of §1 , while the equation

$$Q_0 y = (\operatorname{sgn}\alpha_1(1))\,\left(2\int_0^{Q_0 z} \frac{\xi\, d\xi}{(\alpha_1^2(1) + 2\alpha_2(1)\xi + \xi^2)^{1/2}}\right)^{1/2} \operatorname{sgn}(Q_0 z) \tag{65}$$

is analogous to equation (11). The equations (64) and (65) provide an analytic representation of the manifold Ω_1 .

By substituting the boundary values (63) into (64) we obtain equations for $\alpha_1(1)$ and $\alpha_2(1)$, that is, $1 - \alpha_1(1) = 0$, $-1 - \alpha_2(1) = 0$. Hence

$$\alpha_1(1) = 1 \text{ , } \alpha_2(1) = -1 \quad , \tag{66}$$

and the function $Q_0 x(\tau_1)$ is easily seen to be identically zero in the present case.

The system of equations for $\alpha_1(t)$ and $\alpha_2(t)$ is obtained in the usual way and has the form (cf. (15) in §1)

$$\frac{d\alpha_1}{dt} = 2c\ \alpha_2/\alpha_1 \ , \quad \frac{d\alpha_2}{dt} = 0 \quad .$$

The solution of this system and the supplementary conditions (66) is

$$\alpha_1(t) = (1 + 4c(1-t))^{1/2} \ , \ \alpha_2(t) \equiv -1 \quad .$$

We have for $\Pi_0 x(\tau_0)$ the system

$$\frac{d\Pi_0 z}{d\tau_0} = (\alpha_1(0) + \Pi_0 u_1)\Pi_0 y \ , \quad \frac{d\Pi_0 y}{d\tau_0} = \Pi_0 z \ ,$$
$$\frac{d\Pi_0 u_1}{d\tau_0} = (-1 + \Pi_0 u_2)\Pi_0 y \ , \quad \frac{d\Pi_0 u_2}{d\tau_0} = (\alpha_1(0) + \Pi_0 u_1)\Pi_0 y \tag{67}$$

(for $\alpha_1(0) = (1+4c)^{1/2}$) and the supplementary conditions

$$\Pi_0 z(0) = 1 \ , \ \Pi_0 x(\infty) = 0 \ .$$

The manifold Ω_0 for (67) is represented by

$$\Pi_0 u_1 = -(1+4c)^{1/2} + (1+4c - 2\Pi_0 z + \Pi_0^2 z)^{1/2} \ ,$$
$$\Pi_0 u_2 = \Pi_0 z \ , \tag{68}$$
$$\Pi_0 y = -\left(2 \int_0^{\Pi_0 z} \frac{\xi \, d\xi}{(1+4c-2\xi+\xi^2)^{1/2}}\right)^{1/2} \operatorname{sgn}(\Pi_0 z) \ .$$

It is necessary now to substitute the last equation of (68) into the first equation of (67) and to solve the resulting differential equation for $\Pi_0 z(\tau_0)$ together with the initial condition $\Pi_0 z(0) = 1$. After this the rest of the function $\Pi_0 x(\tau_0)$ is determined from (68) .

The construction of the succeeding terms of the asymptotic expansion can be executed as in §1.

A comparison with experiments shows that that the application of the asymptotic method under consideration is suitable already in the zeroth approximation with a high degree of accuracy in processes involving transistors. A more detailed physical analysis of the mathematical results is given in [14] .

2. Some Control Problems. Certain problems in control theory appear as singularly perturbed boundary value problems of the critical conditionally stable type. As an example, consider the linear problem of Mayer-Bolza [22]

$$\mu \frac{dz}{dt} = A_{11}(t)z + A_{12}(t)y + B_1(t)u \ ,$$

$$\frac{dy}{dt} = A_{21}(t)z + A_{22}(t)y + B_2(t)u \ ,$$

$$z(0,\mu) = z^0 \ , \ y(0,\mu) = y^0 \ ,$$

$$\min\{d^* x(1,\mu) + \frac{1}{2}\int_0^1 [x^*(t,\mu)F(t)x(t,\mu) + u^*(t,\mu)R(t)u(t,\mu)]dt\} \ .$$

Here z and y are M- and m-dimensional phase vectors, respectively, u is the control, x denotes z and y taken together, *denotes transpose, and $F(t)$, $R(t)$ are symmetric matrices.

Suppose that no supplementary conditions involving bounds on the control are imposed. Then the problem becomes a classical problem in the calculus of variations. By applying the method of Lagrange multipliers, we can reduce the problem to the following one for the auxiliary Lagrangean vector functions $\lambda_1(t,\mu)$ and $\lambda_2(t,\mu)$ of dimension M and m , respectively, namely

$$\mu \frac{dz}{dt} = A_{11}z + B_1R^{-1}B_1^*\lambda_1 + A_{12}y + B_1R^{-1}B_2^*\lambda_2 \ ,$$

$$\mu \frac{d\lambda_1}{dt} = F_{11}z - A_{11}^*\lambda_1 + F_{12}y - A_{21}^*\lambda_2 \ ,$$

$$\frac{dy}{dt} = A_{21}z + B_2R^{-1}B_1^*\lambda_1 + A_{22}y + B_2R^{-1}B_2^*\lambda_2 \quad ,$$

$$\frac{d\lambda_2}{dt} = F_{21}z - A_{12}^*\lambda_1 + F_{22}y - A_{22}^*\lambda_2 \quad ,$$

$$z(0,\mu) = z^0 \ , \ \lambda_1(1,\mu) = -d_1/\mu \ , \ y(0,\mu) = y^0 \ , \ \lambda_2(1,\mu) = -d_2 \quad .$$

Here we have denoted by F_{ij} and d_i the appropriate blocks of the matrix F and the vector d.

Thus we obtain a problem with a singular boundary condition at $t = 1$. By changing variables as in Subsection 2 of §2 we are led to a problem having nonsingular boundary conditions and a corresponding matrix with zero eigenvalues. In order to apply our asymptotic methods, it is necessary to assume that the matrix

$$\begin{pmatrix} A_{11} & B_1R^{-1}B_1^* \\ F_{11} & -A_{11}^* \end{pmatrix}$$

has M eigenvalues with negative real parts. Then we obtain a conditionally stable system in the critical case.

Similar kinds of systems occur in other, more complicated problems of optimal control (cf., for example, [1]).

§4 The Case of an Incomplete Set of Eigenvectors

In Chapter 1 we showed for the system of two linear equations (38) that if the number of eigenvectors corresponding to $\lambda \equiv 0$ is less than the multiplicity of this eigenvalue, then in order to obtain a solution bounded as $\mu \to 0$ we must pose, in general, a boundary value problem. Moreover, the asymptotic expansion of the solution will contain fractional powers of μ.

We now consider this question for a certain nonlinear system.

1. A System of Two Nonlinear Equations. Let us consider the system

$$\begin{aligned} \mu \frac{dz}{dt} &= F_1(z,y) + \mu f_1(z,y,t), \\ \mu \frac{dy}{dt} &= F_2(z,y) + \mu f_2(z,y,t), \end{aligned} \qquad 0 \le t \le 1 \tag{69}$$

where the functions $F_i(z,y)$ and $f_i(z,y)$ $(i=1,2)$ are sufficiently smooth for z in (z_1,z_2), y in (y_1,y_2) and t in $[0,1]$.

Suppose that the equation $F_1(z,y) = 0$ has a root $y = \varphi(z)$ in (y_1,y_2) for z in (z_1,z_2) such that $F_{1y}(z,\varphi(z)) \neq 0$, and also that $F_2(z,\varphi(z)) = 0$ for z in (z_1,z_2). Then the reduced system corresponding to (69) has the family of solutions

$$\begin{pmatrix} \bar{z} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi(\alpha) \end{pmatrix}, \ \alpha \text{ in } (z_1,z_2), \tag{70}$$

and

$$\det F_x \equiv \begin{vmatrix} F_{1z}(z,\varphi(z)) & F_{1y}(z,\varphi(z)) \\ F_{2z}(z,\varphi(z)) & F_{2y}(z,\varphi(z)) \end{vmatrix} \equiv 0 \tag{71}$$

for z in (z_1,z_2). In addition, we assume that

$$F_{1z}(z,\varphi(z)) + F_{2y}(z,\varphi(z)) \equiv 0 \text{ in } (z_1,z_2). \tag{72}$$

It follows from (71) and (72) that the matrix F_x has a zero eigenvalue of multiplicity two ($\lambda_1 = \lambda_2 = 0$) to which corresponds the single eigenvector $\binom{1}{\varphi'(\alpha)}$.

Let us introduce new variables z and $w = F_1(z,y)$ in (69), and note that in a neighborhood of $y = \varphi(z)$ we can define a function $y = Y(z,w)$ for which $Y(z,0) = \varphi(z)$. Then the system (69) becomes

$$\begin{aligned} \mu \frac{dz}{dt} &= w + \mu f(z,w,t), \\ \mu \frac{dw}{dt} &= A(z,w)w + \mu g(z,w,t), \end{aligned} \tag{73}$$

where $f(z,w,t) = f_1(z,Y(z,w),t)$, $A(z,w) = (F_{1z} + F_{1y}\frac{F_2}{F_1})(z,Y(z,w))$, and $g(z,w,t) = (F_{1z}f_1 + F_{1y}f_2)(z,Y(z,w),t)$. We have that

$$\lim_{w\to 0} \frac{F_2(z,Y)}{F_1(z,Y)} = \lim_{w\to 0} \frac{F_{2y}(z,Y)}{F_{1y}(z,Y)} = \frac{F_{2y}(z,\varphi(z))}{F_{1y}(z,\varphi(z))},$$

and so

$$A(z,0) = F_{1z}(z,\varphi(z)) + F_{2y}(z,\varphi(z)) = 0$$

by virtue of (72). Thus, the order of $A(z,w)$ as $w\to 0$ is at least w and we can write (73) in the form

$$\begin{aligned} \mu \frac{dz}{dt} &= w + \mu f(z,w,t), \\ \mu \frac{dw}{dt} &= B(z,w)w^2 + \mu g(z,w,t), \end{aligned}$$

where $B(z,0)$ is bounded for z in (z_1,z_2).

Setting $w = \sqrt{\mu}\, v$ we obtain

$$\sqrt{\mu}\,\frac{dz}{dt} = v + \sqrt{\mu}\; f(z,\sqrt{\mu}\, v,t),$$

$$\sqrt{\mu}\,\frac{dv}{dt} = v^2 B(z,\sqrt{\mu}\, v) + g(z,\sqrt{\mu}\, v,t), \tag{74}$$

whose reduced system is

$$\bar{v} = 0$$

$$g(\bar{z},0,t) = 0.$$

We assume that the second of these equations has a root $\bar{z} = \bar{z}(t)$ in (z_1,z_2) and that $g_z(\bar{z},0,t) > 0$ for t in $[0,1]$. The characteristic equation corresponding to

$$\begin{vmatrix} -\Lambda & 1 \\ g_z(\bar{z},0,t) & -\Lambda \end{vmatrix} = 0$$

defines a pair of characteristic values with opposite signs, namely

$$\Lambda_{1,2} = \pm\, (g_z(\bar{z},0,t))^{1/2}.$$

Thus the system (74) is of conditionally stable type with small parameter $\sqrt{\mu}$, to which we can apply the theory developed in [13, Sec. 14].

We consider now various supplementary conditions for the system (69).

1°. Let us first prescribe the initial conditions

$$z(0,\mu) = z^\circ, \ y(0,\mu) = y^\circ, \tag{75}$$

which can be written in terms of the variables z, v as

$$z(0,\mu) = z^\circ, \ v(0,\mu) = \frac{F_1(z^\circ,y^\circ)}{\sqrt{\mu}} .$$

Then, in general, the solution of this initial value problem for the conditionally stable system (74) is unbounded as $\mu \to 0$. (This phenomenon occurs even if the term $1/\sqrt{\mu}$ is absent from the expression for $v(0,\mu)$.)

2°. Let us now prescribe the boundary conditions

$$z(0,\mu) = z^\circ, \ y(1,\mu) = y^1, \tag{76}$$

which we write as

$$z(0,\mu) = z^\circ, \ \sqrt{\mu}\, v(1,\mu) = F_1(z(1,\mu),y^1),$$

in the variables z and v. This boundary value problem, in principle, admits a solution bounded as $\mu \to 0$, and its asymptotic expansion, which features right and left boundary layers, consists of powers of $\sqrt{\mu}$. The question of the existence of such a solution is investigated by the method in [13, Sec. 13] (cf. also [23] and §2 of this chapter).

These remarks also apply to the more general boundary conditions $R(z(0,\mu), z(1,\mu), y(0,\mu), y(1,\mu)) = 0$.

As an illustration of the theory consider the system

$$\mu \frac{dz}{dt} = v + \sqrt{\mu}\, z, \; \mu \frac{dy}{dt} = -(z+y)^2 - (z+y) + \mu(1+t),$$

which can be written as (cf. (74))

$$\sqrt{\mu} \frac{dz}{dt} = v + \sqrt{\mu}\, z, \; \sqrt{\mu} \frac{dv}{dt} = -v^2 + z + 1 + t.$$

The solution of this system satisfying the boundary conditions (76) has the following asymptotic representation valid to order $\mathcal{O}(\sqrt{\mu})$:

$$z = \bar{z} + \Pi_0 z + Q_0 z, \; y = \bar{y} + \Pi_0 y + Q_0 y.$$

Here $\bar{z} = -\bar{y} = -(1+t)$ and $\Pi_0 z = -\Pi_0 y$, $Q_0 z = -Q_0 y$ are found by quadratures from

$$\frac{d}{d\tau_0} \Pi_0 z = -\operatorname{sgn} \Pi_0 z \left(\frac{1}{2} \exp[-2\Pi_0 z] + \Pi_0 z - \frac{1}{2}\right)^{1/2},$$

$$\Pi_0 z(0) = z^0 + 1, \; \tau_0 = t/\sqrt{\mu};$$

$$\frac{d}{d\tau_1} Q_0 z = \operatorname{sgn} Q_0 z \left(\frac{1}{2} \exp[-2Q_0 z] + Q_0 z - \frac{1}{2}\right)^{1/2},$$

$$Q_0 z(0) = 2 - y^1, \; \tau_1 = (1-t)/\sqrt{\mu}.$$

2. A Problem Arising in the Theory of Singular Optimal Control. [This problem has been investigated by M. G. Dimitriev.] Suppose that it is required to minimize the functional

$$J = \varphi(x_1(1)) \tag{77}$$

along trajectories of the system of two equations

$$\frac{dx_1}{dt} = f_1(x_1,x_2,t), \qquad 0 \le t \le 1$$

$$\frac{dx_2}{dt} = f_2(x_1,x_2,t) + f_3(x_2,t)u, \tag{78}$$

$$x_1(0) = x_1^o, \; x_2(0) = x_2^o.$$

There is no bound on the control u of the type involving a closure inequality. Now this problem need not have a solution in the class of continuous functions u and so there is the question of the construction of a generalized solution of the problem (77), (78).

To this end we introduce a regularized problem, that is, in place of the functional (77) we consider the functional

$$J_\mu = \varphi(x_1(1,\mu)) + \frac{\mu}{2}\int_0^1 u^2 dt.$$

If we then introduce conjugate variables (Lagrange multipliers), we can reduce this problem, as in Section 3 of the present chapter, to a system of differential equations with boundary conditions, namely

$$\frac{dx_1}{dt} = f_1(x_1,x_2,t),$$

$$\mu\frac{dx_2}{dt} = \mu f_2(x_1,x_2,t) + f_3^2(x_2,t)\psi_2, \tag{80}$$

$$\frac{d\psi_1}{dt} = -(f_{1x_1}\psi_1 + f_{2x_1}\psi_2),$$

$$\mu \frac{d\psi_2}{dt} = -\mu(f_{1x_2}\psi_1 + f_{2x_2}\psi_2) - f_{3x_2}f_3\psi_2^2,$$

$$x_1(0,\mu) = x_1^o, \; x_2(0,\mu) = x_2^o, \tag{81}$$

$$\psi_1(1,\mu) = -\varphi_{x_1}(x_1(1,\mu)), \; \psi_2(1,\mu) = 0.$$

The right-hand sides of the equations for the fast variables x_2 and ψ_2 have the same properties as in the system (69). (The presence of the slow variables x_1 and ψ_1 offers no complication; cf. Section 5 of Chapter 2.) In fact, if we assume that $f_3(x_2,t) \neq 0$, then setting $\mu = 0$ in the second and the fourth equations of (80) we obtain $\psi_2 = 0$, while x_2 remains undetermined. The determinant (71) here has the form

$$\begin{vmatrix} 0 & f_3^2 \\ 0 & 0 \end{vmatrix}$$

and consequently, $\lambda_1 = \lambda_2 = 0$; thus condition (72) holds.

The change of variable described in Subsection 1, having the form $v = f_3^2\psi_2/\sqrt{\mu}$, leads to a problem in which the characteristic values are again equal to zero. However, by using the change of variable $v = f_3\psi_2/\sqrt{\mu}$ we obtain in place of (80), (81) the following problem

$$\frac{dx_1}{dt} = f_1, \frac{d\psi_1}{dt} = -f_{1x_1}\psi_1 - \sqrt{\mu}\,\frac{f_{2x_1}}{f_3}v,$$

$$\sqrt{\mu}\frac{dx_2}{dt} = \sqrt{\mu}\,f_2 + f_3 v, \tag{82}$$

$$\sqrt{\mu}\frac{dy}{dt} = \sqrt{\mu}\left(\frac{f_2 f_{3x_2} + f_{3t}}{f_3} - f_{2x_2}\right)v - f_{1x_2}f_3\psi_1,$$

$$x_1(0,\mu) = x_1^o,\ x_2(0,\mu) = x_2^o,$$

$$\psi_1(1,\mu) = -\varphi_{x_1}(x_1(1,\mu)),\ v(1,\mu) = 0. \tag{83}$$

This problem is conditionally stable if we assume that

$$f_{1x_2x_2}(\bar{x}_{10},\bar{x}_{20},t)\bar{\psi}_{10} < 0. \tag{84}$$

An investigation of this problem by the methods discussed in Subsection 1 reveals that the leading term of the asymptotic expansion for $x_2(t,\mu)$ has a boundary layer at the left endpoint. The asymptotic expansion of the control $u(t,\mu)$ to order $\Theta(\sqrt{\mu})$ is found from the formula $u = v/\sqrt{\mu}$, once we have found the expansion of v to order $\Theta(\mu)$. After the corresponding calculations have been made, we find that to order $\Theta(\sqrt{\mu})$ the control $u(t,\mu)$ has the form

$$u(t,\mu) = \frac{\Pi_0 v(\tau_0)}{\sqrt{\mu}} + \bar{v}_1(t) + \Pi_1 v(\tau_0) + Q_1 v(\tau_1), \tag{85}$$

while the optimal trajectory is

$$x_1(t,\mu) = \bar{x}_{10}(t), \quad x_2(t,\mu) = \bar{x}_{20}(t) + \Pi_0 x_2(\tau_0). \tag{86}$$

From the relation (85) it follows that the leading term of the asymptotic expansion as $\mu \to 0$ has the character of a δ-function.

Remarks. 1. Using singular perturbation theory, we can define a class of singular functions in which the problem (77), (78) is solvable.

2. The method discussed above can be applied in the vector case. For example, if u and x_2 are k-dimensional vectors and f_3 a $(k \times k)$-matrix, then the problem of zero characteristic values can be eliminated by means of the change of variable $v = f_3^* \psi_2 / \sqrt{\mu}$.

Chapter 4

Singularly Perturbed Integro-differential Equations in the Critical Case

§1 Statement of the Problem and Auxiliary Results

1. Statement of the Problem. The results of Chapter 1 can be extended in a natural way to cases where $A(t)$ is no longer a matrix, but rather a more complex linear operator. In this chapter such a generalization is made in the following direction: x is assumed to be a scalar function of two variables t and s as well as of the parameter μ, while A is assumed to be an integral operator, integration being with respect to s. (Analogous problems in a more abstract form are considered in [28].)

Thus, we consider the equation

$$\mu \frac{\partial x(t,s,\mu)}{\partial t} = -[x(t,s,\mu) - \int_a^b K(s,\sigma)x(t,\sigma,\mu)d\sigma] + \mu f(x,t,s,\mu)\,(0 \le t \le T,\ a \le s \le b)\ , \tag{1}$$

which can be written in the simple form

$$\mu \frac{\partial x}{\partial t} = Ax + \mu f \equiv -[x - Bx] + \mu f\ , \tag{1'}$$

and we prescribe the initial condition

$$x(0,s,\mu) = x^0(s)\ . \tag{2}$$

Let us assume that $\lambda = 1$ is an eigenvalue of the operator B. Then the operator A has the eigenvalue $\lambda = 0$, and so the reduced equation $\bar{A}\bar{x} = 0$ has a family of solutions which depends on one or more arbitrary functions of t. The same questions arise here as in the previous chapters: To which member of this family does the solution of the problem (1), (2) converge as $\mu \to 0$, that is, how do we determine the function of t in the family of solutions for the reduced equation which provides the limiting solution of (1), (2)? What does the asymptotic expansion of the solution of this problem with respect to μ look like?

Suppose that the following conditions are satisfied:

I. $K(s,\sigma)$ is continuous in the square $R = \{a \leq s \leq b\ ,\ a \leq \sigma \leq b\}$, $f(x,t,s,\mu)$ is continuous with respect to s and sufficiently smooth with respect to x,t and μ in a domain $D(x,t,s,\mu) = D(x,t,s) \times [0,\mu_0]$, where $D(x,t,s)$ is some domain in the space of the variables (x,t,s), and $x^0(s)$ is continuous for $a \leq s \leq b$.

II. The kernel $K(s,\sigma)$ is symmetric, that is, $K(s,\sigma) = K(\sigma,s)$ in R.

III. The eigenvalues of the operator B are such that

$$\lambda_1 = \dots = \lambda_k = 1\ ,\ \lambda_i < 1 \quad \text{for} \quad i = k + 1, \dots \ .$$

The corresponding set of eigenfunctions $\{\varphi_i(s)\}$ is assumed to be orthonormal, that is,

$$\int_a^b \varphi_i(s)\varphi_j(s)\,ds = \delta_{ij} = \begin{cases} 1, & i = j\ , \\ 0, & i \neq j\ . \end{cases}$$

In what follows we will denote the scalar product of two functions $u(s)$ and $v(s)$ by

$$\langle u,v\rangle = \langle u(s),v(s)\rangle = \int_a^b u(s)v(s)\,ds\ .$$

2. <u>An Auxiliary Lemma</u>. Let us consider an equation of the form

$$\frac{\partial y(\tau,s)}{\partial\tau} = -\,[y(\tau,s) - \int_a^b K(s,\sigma)y(\tau,\sigma)\,d\sigma] + g(\tau,s) \tag{3}$$

$$(\tau \geq 0\ ,\ a \leq s \leq b)$$

or

$$\frac{\partial y}{\partial\tau} = -[y - By] + g \tag{3'}$$

together with the initial condition

$$y(0,s) = y^0(s)\ . \tag{4}$$

(As we will see below, this is an equation for a boundary function.)

We make the following assumptions:

1^0. $y^0(s)$ is continuous for $a \leq s \leq b$.

2^0. The operator B satisfies Conditions I-III.

3^0. $g(\tau,s)$ is continuous for $\tau \geq 0$ and $a \leq s \leq b$, and satisfies the estimate $|g(\tau,s)| \leq c\exp(-\kappa\tau)$, where c and k are positive constants.

We now pose the following questions: How does one choose $y^0(s)$ in order that the solution of the problem (3), (4) will converge to zero as $\tau \to \infty$? Will this convergence be of exponential type, that is, do we have an estimate of the same type as that for g in 3^0 ? The answers to these questions are given by

Lemma 1. Suppose that Conditions $1^0 - 3^0$ are satisfied, and that the function $y^0(s)$ is such that

$$\langle y_0(s), \varphi_i(s)\rangle = -\int_0^\infty \langle g(\tau,s), \varphi_i(s)\rangle d\tau \quad (i = 1,\dots k). \tag{5}$$

Then the solution $y(\tau,s)$ of the problem (3), (4) exists and is continuous for $\tau \geq 0$ and $a \leq s \leq b$, and converges to zero as $\tau \to \infty$ uniformly for s in $[a,b]$; moreover,

$$|y(\tau,s)| \leq c\exp(-\varkappa\tau) \quad . \tag{6}$$

(We note, as in previous chapters, that the constants c and $\varkappa$ in various estimates of the type (6) are not generally the same, despite the fact that they are denoted by the same letters.)

Proof. It is not difficult to prove the existence and the continuity of the solution in the stated domain if we pass from (3), (4) to the integral equation

$$y(\tau,s) = y^0(s)\exp(-\tau) + \int_0^\tau \exp(-(\tau-\theta)) \times \\ \times \left[\int_a^b K(s,\sigma)y(\theta,\sigma)d\sigma + g(\theta,s)\right]d\theta \tag{7}$$

and apply the method of successive approximations. For this to succeed we require only the continuity of $y^0(s)$, $K(s,\sigma)$ and $g(\tau,s)$, and the boundedness of $g(\tau,s)$.

Let us then establish the estimate (6). We set

$$y(\tau,s) = \delta_1(\tau,s) + \delta_2(\tau,s) \quad , \tag{8}$$

where

$$\delta_2(\tau,s) = \sum_{i=1}^{k} \delta_{2i}(\tau)\varphi_i(s), \ \delta_{2i}(\tau) = \langle y(\tau,s), \varphi_i(s)\rangle \quad .$$

Hence, it follows that

$$\langle\delta_1(\tau,s), \varphi_i(s)\rangle = 0 \ , \ \langle B\delta_1,\varphi_i\rangle = 0 \quad (i = 1,\ldots,k) \ . \tag{9}$$

Substituting (8) into (3') we obtain

$$\frac{\partial\delta_1}{\partial\tau} + \frac{\partial\delta_2}{\partial\tau} = - [\delta_1 + \delta_2 - B\delta_1 - B\delta_2] + g \quad ,$$

and since $\delta_2 - B\delta_2 = 0$, that

$$\frac{\partial\delta_1}{d\tau} + \frac{\partial\delta_2}{\partial\tau} = - [\delta_1 - B\delta_1] + g \ . \tag{10}$$

After taking the scalar product of (10) with $\varphi_i(s)$ $(i = 1,\ldots,k)$ and using (9) we obtain

$$\frac{d\delta_{2i}}{d\tau} = \langle g,\varphi_i\rangle \quad (i = 1,\ldots,k) \quad . \tag{11}$$

By virtue of (5)

$$\delta_{2i}(0) = \langle y^0,\varphi_i\rangle = - \int_0^\infty \langle g,\varphi_i\rangle d\tau \quad ,$$

and therefore it follows from (11) that

$$\delta_{2i}(\tau) = -\int_0^\infty \langle g,\varphi_i\rangle d\tau + \int_0^\tau \langle g,\varphi_i\rangle d\tau = -\int_\tau^\infty \langle g,\varphi_i\rangle d\tau .$$

In view of Condition 3^0 we have that $\delta_{2i}(\tau)$, and consequently, $\delta_2(\tau,s)$ satisfy an exponential estimate, namely

$$|\delta_{2i}(\tau)| \leq c\exp(-\varkappa\tau) , \quad |\delta_2(\tau,s)| \leq c\exp(-\varkappa\tau)$$
$$(\tau \geq 0 , \; a \leq s \leq b) . \tag{12}$$

From (10) we obtain now for δ_1 the equation

$$\frac{\partial\delta_1}{\partial\tau} = -[\delta_1 - B\delta_1] + g_1 , \tag{13}$$

where

$$g_1 = g - \sum_{i=1}^{k} \frac{d\delta_{2i}}{d\tau}\varphi_i = g - \sum_{i=1}^{k} \langle g,\varphi_i\rangle\varphi_i \; ; \; |g_1| \leq c\exp(-\varkappa\tau) .$$

Taking the scalar product of (13) with $2\delta_1$, we obtain

$$\frac{d}{d\tau}\langle\delta_1,\delta_1\rangle = -2[\langle\delta_1,\delta_1\rangle - \langle\delta_1,B\delta_1\rangle] + 2\langle\delta_1,g_1\rangle . \tag{14}$$

By the Hilbert-Schmidt Theorem $B\delta_1 = \sum_{i=1}^{\infty} \lambda_i < \delta_1, \varphi_i > \varphi_i$, where the summation begins with $i = k+1$ because of (9). Hence,

$$\langle\delta_1,B\delta_1\rangle = \sum_{i=k+1}^{\infty} \lambda_i \langle\delta_1,\varphi_i\rangle^2 . \tag{15}$$

Since $\lambda_i < 1$ for $i = k+1,\ldots$ and since $\lambda = 1$ is not a limit point of the spectrum of B, there exists a positive constant $\bar{\lambda} < 1$ such that $\lambda_i \leq \bar{\lambda}$ for $i = k+1,\ldots$. Further, Bessel's inequality

$\sum_{i=k+1}^{\infty} \langle \delta_1, \varphi_i \rangle^2 \leq \langle \delta_1, \delta_1 \rangle$ implies that

$$\langle \delta_1, B\delta_1 \rangle \leq \bar{\lambda} \langle \delta_1, \delta_1 \rangle ;$$

so

$$\langle \delta_1, \delta_1 \rangle - \langle \delta_1, B\delta_1 \rangle \geq (1-\bar{\lambda}) \langle \delta_1, \delta_1 \rangle \quad , \tag{16}$$

where $\bar{\lambda} < 1$.

Let us rewrite equation (14) as

$$\frac{d}{d\tau}\langle \delta_1, \delta_1 \rangle = -2D(\tau)\langle \delta_1, \delta_1 \rangle + 2\langle \delta_1, g_1 \rangle \quad , \tag{17}$$

where

$$D(\tau) = \frac{\langle \delta_1, \delta_1 \rangle - \langle \delta_1, B\delta_1 \rangle}{\langle \delta_1, \delta_1 \rangle} \geq 1 - \bar{\lambda} > 0 \; . \tag{18}$$

Dividing by $2\langle \delta_1, \delta_1 \rangle^{1/2}$

$$\frac{d}{d\tau}\langle \delta_1, \delta_1 \rangle^{1/2} = -D(\tau)\langle \delta_1, \delta_1 \rangle^{1/2} + \frac{\langle \delta_1, g_1 \rangle}{\langle \delta_1, \delta_1 \rangle^{1/2}} \quad ,$$

and integrating, we obtain

$$\begin{aligned} \langle \delta_1, \delta_1 \rangle^{1/2} = \langle \delta_1(0,s), \delta_1(0,s) \rangle^{1/2} \exp(-\int_0^{\tau} D(\theta)\, d\theta) \; + \\ \int_0^{\tau} \exp(-\int_{\theta}^{\tau} D(\theta)\, d\theta) \langle \delta_1, g_1 \rangle \langle \delta_1, \delta_1 \rangle^{-1/2} d\theta \; . \end{aligned} \tag{19}$$

By virtue of Cauchy's inequality and Condition 3^0

$$|\langle \delta_1, g_1 \rangle| \leq \langle \delta_1, \delta_1 \rangle^{1/2} \langle g_1, g_1 \rangle^{1/2} \leq \langle \delta_1, \delta_1 \rangle^{1/2} \exp(-\varkappa\tau) \quad .$$

Thus

$$\langle \delta_1, \delta_1 \rangle^{1/2} \leq c \exp(-\varkappa\tau) \; (\tau \geq 0, \; a \leq s \leq b) \quad .$$

In order to obtain a similar estimate for δ_1 itself, we note that from (13) $\delta_1(\tau,s)$ can be represented by

$$\delta_1(\tau,s) = \delta_1(0,s)\exp(-\tau) + \int_0^\tau \exp(-(\tau-\theta))\left[\int_a^b K(s,\sigma)\,\times\right.$$

$$\left.\times\,\delta_1(\theta,\sigma)d\sigma + g_1(\theta,s)\right]d\theta \quad .$$

So

$$|\delta_1(\tau,s)| \le c\exp(-\varkappa\tau)\ (\tau \ge 0,\ a \le s \le b) \quad ,$$

follows since

$$\left|\int_a^b K(s,\sigma)\delta_1(\theta,s)d\sigma\right| \le \langle K,K\rangle^{1/2}\langle\delta_1,\delta_1\rangle^{1/2} \le c\exp(-\varkappa\theta) \quad .$$

Thus, from (8) and (12) we finally have the estimate

$$|y(\tau,s)| \le c\exp(-\varkappa\tau)\ (\tau \ge 0,\ a \le s \le b) \quad .$$

This concludes the proof of Lemma 1.

§2 Construction of the Asymptotic Expansion

1. <u>An Algorithm for the Construction of the Expansion</u>. The asymptotic expansion of the solution of problem (1), (2) in the parameter μ is sought in the usual form

$$\begin{aligned} x(t,s,\mu) &= \bar{x}(t,s,\mu) + \Pi x(\tau,s,\mu) \\ &= \bar{x}_0(t,s) + \mu\bar{x}_1(t,s) + \ldots + \mu^n\bar{x}_n(t,s) + \ldots + \\ &\quad + \Pi_0x(\tau,s) + \mu\Pi_1x(\tau,s) + \ldots + \mu^n\Pi_nx(\tau,s) + \ldots \\ &\qquad (\tau = t/\mu) \;. \end{aligned} \tag{20}$$

By substituting (20) into (3), (4) and equating coefficients of like powers of μ, separately for coefficients depending on (t,s) and those depending on (τ,s), we obtain equations and supplementary conditions for the determination of $\bar{x}_i(t,s)$ and $\Pi_i x(\tau,s)$. By substituting (20) into the nonlinear function $f(x,t,s,\mu)$ we can express f as $\bar{f} + \Pi f$, just as in previous chapters.

We obtain first the equation

$$\bar{x}_0(t,s) = \int_a^b K(s,\sigma)\bar{x}_0(t,\sigma)\,d\sigma \quad .$$

Bu virtue of Condition III its general solution is

$$\bar{x}_0(t,s) = \sum_{i=1}^{k} \alpha_i(t)\varphi_i(s) \quad , \tag{21}$$

where the $\alpha_i(t)$ are as yet arbitrary functions.

The equation and the initial condition for $\Pi x_0(\tau,s)$ are

$$\frac{\partial \Pi_0 x(\tau,s)}{\partial \tau} = - \left[\Pi_0 x(\tau,s) - \int_a^b K(s,\sigma)\Pi_0 x(\tau,\sigma)\,d\sigma\right] \quad ,$$

$$\Pi_0 x(0,s) = x^0(s) - \bar{x}_0(0,s) = x^0(s) - \sum_{i=0}^{k} \alpha_i(0)\varphi_i(s) \quad .$$

In addition, as is our custom, we impose the restriction that

$$\Pi_0 x(\tau,s) \to 0 \quad \text{as} \quad \tau \to \infty \quad . \tag{22}$$

Thus we have for $\Pi_0 x(\tau,s)$ a problem of the type considered in Subsection 2 of §1 for the case $g(\tau,s) = 0$. By virtue of Lemma 1, $\Pi_0 x(\tau,s)$ satisfies the condition in (22), and moreover, the inequality

$$|\Pi_0 x(\tau,s)| \le c\exp(-\kappa\tau)\ (\tau \ge 0,\ a \le s \le b)\ , \tag{23}$$

provided the initial value $\Pi_0 x(0,s)$ satisfies the condition in (5), that is,

$$\langle x^0(s) - \sum_{j=1}^{k} \alpha_j(0)\varphi_j(s), \varphi_i(s)\rangle = 0 \quad (i = 1,\ldots,k). \tag{24}$$

Hence,

$$\alpha_i(0) = \langle x^0(s), \varphi_i(s)\rangle \quad (i = 1,\ldots,k)\ . \tag{25}$$

As was the case in previous chapters, we now determine the functions $\alpha_i(t)$ completely by considering the equation for $\bar{x}_1(t,s)$, namely

$$\frac{\partial \bar{x}_0(t,s)}{\partial t} = -\left[\bar{x}_1(t,s) - \int_a^b K(s,\sigma)\bar{x}_1(t,\sigma)\,d\sigma\right] + f(\bar{x}_0(t,s),t,s,0)$$

or

$$\bar{x}_1(t,s) - \int_a^b K(s,\sigma)\bar{x}_1(t,\sigma)\,d\sigma = f\left(\sum_{i=1}^{k} \alpha_i(t)\varphi_i(s),t,s,0\right) - \sum_{i=1}^{k} \frac{d\alpha_i(t)}{dt}\varphi_i(s) \equiv \psi(t,s)\ . \tag{26}$$

For the solvability of this inhomogeneous problem it is necessary and sufficient that the right-hand side $\psi(t,s)$ be orthogonal to each $\varphi_i(s)$ $(i = 1,\ldots,k)$. This orthogonality condition is itself represented by a system of differential equations for $\alpha_i(t)$, namely

$$\frac{d\alpha_i}{dt} = \langle f\left(\sum_{j=1}^{k} \alpha_j(t)\varphi_j(s),t,s,0\right), \varphi_i(s)\rangle \quad (i = 1,\ldots,k)\ . \tag{27}$$

IV. Suppose that the system (27) together with the initial conditions (25) has the solutions $\alpha_i = \alpha_i(t)$ for $0 \leq t \leq T$.

Thus the function $\bar{x}_0(t,s)$ is determined completely by the formula (21), and the construction of the zeroth term in the expansion is finished.

Let us now introduce in the space of the variables (x,t,s) a surface L which consists of the two parts:

$$L_1 = \{(x,t,s):\ x = \bar{x}_0(0,s) + \Pi_0 x(\tau,s)(\tau \geq 0)\ ;\ t = 0;\ a \leq s \leq b\}\ ,$$

$$L_2 = \{(x,t,s):\ x = \bar{x}_0(t,s);\ 0 \leq t \leq T;\ a \leq s \leq b\}\ .$$

It is natural to require that the following condition holds:

V. Suppose that the surface L belongs to the domain $D(x,t,s)$ which appears in Condition I.

The general solution of equation (26) can be written as

$$\bar{x}_1(t,s) = \sum_{i=1}^{k} \beta_i(t)\varphi_i(s) + \tilde{x}_1(t,s)\ , \tag{28}$$

where the $\beta_i(t)$ are as yet arbitrary functions and $\tilde{x}_1(t,s)$ is a particular solution of (26) which, for example, has the form

$$\tilde{x}_1(t,s) = \psi(t,s) + \sum_{i=k+1}^{\infty} \frac{\lambda_i}{1-\lambda_i}\psi_i(t)\varphi_i(s), \psi_i(t) = \langle\psi(t,s),\varphi_i(s)\rangle\ .$$

The equation and the supplementary conditions for $\Pi_1 x(\tau,s)$ are

$$\frac{\partial\Pi_1 x(\tau,s)}{\partial\tau} = -\ [\Pi_1 x(\tau,s) - \int_a^b K(s,\sigma)\Pi_1 x(\tau,\sigma)d\sigma] + g(\tau,s)\ ,$$

where

$$g(\tau,s) = \Pi_0 f = f(\bar{x}_0(0,s) + \Pi_0 x(\tau,s),0,s,0) - f(\bar{x}_0(0,s),0,s,0) \ ,$$

$$\Pi_1 x(0,s) = -\bar{x}_1(0,s) \ ,$$

$$\Pi_1 x(\tau,s) \to 0 \quad \text{as} \quad \tau \to \infty \ . \tag{29}$$

By virtue of (23) the function $g(\tau,s)$ satisfies the exponential estimate $|g(\tau,s)| \le c\exp(-\varkappa\tau)$ $(\tau \ge 0,\ a \le s \le b)$. Thus the function $\Pi_1 x(\tau,s)$ is the solution of a problem of the type considered in Subsection 2 of §1 . And by virtue of Lemma 1 it satisfies condition (29) and an exponential estimate, provided that

$$\langle -\bar{x}_1(0,s),\varphi_i(s)\rangle = -\int_0^\infty \langle g(\tau,s),\varphi_i(s)\rangle d\tau \quad (i = 1,\ldots,k) \ .$$

By inserting here the expression (28) for $\bar{x}_1(0,s)$, we obtain the values $\beta_i(0)$, namely

$$\beta_i(0) = \int_0^\infty \langle g(\tau,s),\varphi_i(s)\rangle d\tau \quad (i = 1,\ldots,k) \ .$$

Thus, $\Pi_1 x(\tau,s)$ is completely determined, and we have found the initial values $\beta_i(0)$. The functions $\beta_i(t)$ are determined completely from a solvability condition for the integral equation defining $\bar{x}_2(t,s)$, in a manner analogous to that for the determination of the functions $\alpha_i(t)$. It turns out that the $\beta_i(t)$ satisfy the system of linear differential equations

$$\frac{d\beta_i}{dt} = \sum_{j=1}^{k} b_{ij}(t)\beta_j(t) + f_1(t) \quad (i = 1,\ldots,k) \ , \tag{30}$$

where $b_{ij}(t) = \langle f_x(\bar{x}_0(t,s),t,s,0)\varphi_i(s),\varphi_j(s)\rangle$ and $f_1(t)$ is a known function.

The determination of the remaining terms in the expansion proceeds according to considerations analogous to those used for the determination of $\bar{x}_1(t,s)$ and $\Pi_1 x(\tau,s)$. At the i-th step the expression for $\bar{x}_i(t,s)$ contains k arbitrary functions (let us denote them by $\gamma_i(t)$, $i=1,\ldots,k$). Now the problem for $\Pi_i x(\tau,s)$ is analogous to the one for $\Pi_1 x(\tau,s)$, while the values $\gamma_i(0)$ are found from a condition like (5). Finally, we obtain a system of linear differential equations like (30) for $\gamma_i(t)$ from a solvability condition in the equation for $\bar{x}_{i+1}(t,s)$. It follows also that each Π-function satisfies an exponential estimate

$$|\Pi_i x(\tau,s)| \leq c \exp(-\kappa\tau) \ (\tau \geq 0, \ a \leq s \leq b) \quad .$$

2. <u>An Estimate of the Remainder Term</u>. Let us first make more precise the requirement involving the smoothness of $f(x,t,s,\mu)$ (cf. I). It is possible to take as the domain $D(x,t,s)$ occurring in Condition I an arbitrary δ-tube of the surface L (cf. V). We then require that $f(x,t,s,\mu)$ have continuous partial derivatives up to order $n+2$ inclusive with respect to x,t and μ in the domain $D(x,t,s,\mu) = D(x,t,s,\mu) \times [0,\mu_0]$. We have determined the terms of the series (20) to order $n+1$ inclusive, and let us denote by $X_k(t,s,\mu)$ the k-th partial sum of (20), that is,

$$X_k(t,s,\mu) = \sum_{i=0}^{k} \mu^i(\bar{x}_i(t,s) + \Pi_i x(\tau,s)) \quad .$$

Let us now introduce the norm of a function $y(t,s,\mu)$ by $\|y(t,s,\mu)\| = \sup\{|y(t,s,\mu)| : \ 0 \leq t \leq T, \ a \leq s \leq b\}$.

Theorem 4. Under Conditions I-V there exist positive constants μ_0 and c such that for $0 < \mu \le \mu_0$ the solution $x(t,s,\mu)$ of the problem (1), (2) exists in the domain $\{0 \le t \le T,\ a \le s \le b\}$, is unique and satisfies the inequality

$$\|x(t,s,\mu) - X_n(t,s,\mu)\| \le c\mu^{n+1} .$$

Proof. (Suggested by A. Kasanov.) Set $\xi(t,s,\mu) = x(t,s,\mu) - X_{n+1}(t,s,\mu)$. Then by substituting $x = X_{n+1} + \xi$ into (1), (2) we obtain for ξ the problem

$$\mu\frac{\partial\xi}{\partial t} = -[\xi - B\xi] + \mu f_x(t,s,\mu)\xi + G(\xi,t,s,\mu) , \qquad (31)$$

$$\xi(0,s,\mu) = 0 , \qquad (32)$$

where

$$f_x(t,s,\mu) = f_x(\bar{x}_0(t,s) + \Pi_0 x(\tau,s),t,s,\mu) \text{ and}$$

$$G(\xi,t,s,\mu) = -[X_{n+1} - BX_{n+1}] + \mu f(X_{n+1} + \xi,t,s,\mu)$$
$$- \mu f_x(t,s,\mu)\xi - \mu\frac{dX_{n+1}(t,s,\mu)}{dt} .$$

As in the previous chapters, it is not difficult to show that $G(\xi,t,s,\mu)$ has the following two properties:

1. $\|G(0,t,s,\mu)\| = O(\mu^{n+2})$.

2. $G(\xi,t,s,\mu)$ is a contraction operator with contraction coefficient of order $\mathcal{O}(\mu^2)$ for $\|\xi\| = \mathcal{O}(\mu)$. This means that if $\|\xi_1(t,s,\mu)\| \le c_1\mu$ and $\|\xi_2(t,s,\mu)\| \le c_1\mu$, then there are constants c_0 and $\mu_0 \le \mu_1$ such that for $0 < \mu \le \mu_0$

$$\|G(\xi_1,t,s,\mu) - G(\xi_2,t,s,\mu)\| \le c_0\mu^2\|\xi_1 - \xi_2\| \ . \tag{33}$$

See our remark in Subsection 3, §1 of Chapter 1 regarding the constant μ_0.

Let us now write (31) as

$$U\xi = G(\xi,t,s,\mu) \ , \tag{34}$$

where U is the operator defined by $U\xi = \mu\frac{\partial\xi}{\partial t} + [\xi - B\xi] - \mu f_x\xi$.

We consider first the auxiliary linear problem

$$Uy = \alpha(t,s,\mu), \ y(0,s,\mu) = 0 \ , \tag{35}$$

where $\alpha(t,s,\mu)$ is a given continuous function.

<u>Lemma 2</u>. For each sufficiently small value of $\mu (0 < \mu \le \mu_0)$ the solution $y(t,s,\mu)$ of problem (35) exists in the domain $\{0 \le t \le T, a \le s \le b\}$, is unique, continuous and satisfies the inequality

$$\|y(t,s,\mu)\| \le M\mu^{-1}\|\alpha(t,s,\mu)\| \ , \tag{36}$$

where the constant $M > 0$ is independent of μ.

Remark. The constant in this estimate, because of its importance for later discussion, is denoted by M rather than by c .

The existence, uniqueness and continuity of $y(t,s,\mu)$ can be established without difficulty by passing from (35) to the integral equation

$$y(t,s,\mu) = \int_0^t \exp(\mu^{-1}\int_\theta^t (-1+\mu f_x)\,d\theta)\,\frac{By+\alpha(\theta,s,\mu)}{\mu}\,d\theta \tag{37}$$

and applying the method of successive approximations.

Let us prove that the estimate (36) is valid. By taking the scalar product of equation (35) with y we obtain the equation

$$\frac{1}{2}\mu\frac{d}{dt}\langle y,y\rangle = -\langle y,y\rangle + P\langle y,y\rangle + \mu Q\langle y,y\rangle + R\langle y,y\rangle^{1/2} , \tag{38}$$

where

$$P = \frac{\langle y,By\rangle}{\langle y,y\rangle} , \quad Q = \frac{\langle y,f_x y\rangle}{\langle y,y\rangle} , \quad R = \frac{\langle y,\alpha\rangle}{\langle y,y\rangle^{1/2}} .$$

We know already from an analogous situation that

$$\langle y,By\rangle = \langle y, \sum_{i=1}^{\infty} \lambda_i\langle y,\varphi_i\rangle\varphi_i\rangle = \sum_{i=1}^{\infty} \lambda_i\langle y,\varphi_i\rangle^2$$
$$\leq \sum_{i=1}^{\infty}\langle y,\varphi_i\rangle^2 \leq \langle y,y\rangle ,$$

so $P \leq 1$. Moreover, it is clear that

$$\|Q\| \leq \|f_x\| \text{ and } \|R\| \leq \|\alpha\|(b-a)^{1/2} . \tag{39}$$

Dividing (38) by $\langle y,y\rangle^{1/2}$ we obtain

$$\mu\frac{d}{dt}\langle y,y\rangle^{1/2} = (-1+P+\mu Q)\langle y,y\rangle^{1/2} + R \quad ,$$

and so

$$\langle y,y\rangle^{1/2} = \int_0^t \exp(\mu^{-1}\int_\theta^t(-1+P+\mu Q)\,d\theta)\cdot\frac{R}{\mu}\,d\theta \quad .$$

By virtue of the properties just noted P and Q are exponentially bounded, and consequently, by taking account of the inequality in (39) for R, we have the estimate

$$\langle y,y\rangle^{1/2} \le c\mu^{-1}\|\alpha\| \quad . \tag{40}$$

From the obvious inequality

$$\exp(\mu^{-1}\int_\theta^t(-1+\mu f_x)\,d\theta) \le \exp(-\mu^{-1}\varkappa(t-\theta)) \quad ,$$

(37) implies now that

$$\|y(t,s,\mu)\| \le c(\|By\| + \|\alpha\|) \quad . \tag{41}$$

Since

$$|By| = |\int_a^b K(s,\sigma)y(t,\sigma,\mu)\,d\sigma| \le \langle K,K\rangle^{1/2}\langle y,y\rangle^{1/2} \le \frac{c}{\mu}\|\alpha\| \quad ,$$

it follows from (40) that $\|By\| \le \frac{c}{\mu}\|\alpha\|$, and so,

$$\|y\| \le \frac{c}{\mu}\|\alpha\| \quad .$$

If we replace the constant c by M then the lemma is proved.

We turn now to equation (34) and the initial condition (32). We will prove the existence of a unique solution satisfying the estimate

$$\|\xi\| = \Theta(\mu^{n+1}) \tag{42}$$

for all sufficiently small μ. This will establish the theorem.

In order to apply the method of successive approximations we set $\xi_0 = 0$ and define ξ_{k+1} inductively by $U\xi_{k+1} = G(\xi_k,t,s,\mu)$. By virtue of Property 1 for G $\|G(\xi_0,t,s,\mu)\| \leq c\mu^{n+2}$, while from Lemma 2

$$\|\xi_1\| \leq \frac{M}{\mu} c\mu^{n+2} = Mc\mu^{n+1} \equiv \frac{c_1}{2}\mu^{n+1} \quad . \tag{43}$$

We continue now by induction on indices knowing that $\|\xi_0\| \leq c_1\mu^{n+1}$ and $\|\xi_1\| \leq c_1\mu^{n+1}$. If we assume that

$$\|\xi_i\| \leq c_1\mu^{n+1} \quad (i = 0,1,\dots k) \ , \tag{44}$$

then we must prove that (44) is also valid for $i = k+1$. Clearly

$$U(\xi_{k+1} - \xi_k) = G(\xi_k,t,s,\mu) - G(\xi_{k-1},t,s,\mu) \ ,$$

and since $\|\xi_k\| \leq c_1\mu^{n+1} \leq c_1\mu$ and $\|\xi_{k-1}\| \leq c_1\mu$ for $0 < \mu \leq \mu_1 \leq 1$, we can apply the inequality (33) to the difference in the right-hand side. In addition, by using Lemma 2, we obtain

$$\|\xi_{k+1} - \xi_k\| \leq \frac{M}{\mu} c_0\mu^2\|\xi_k - \xi_{k-1}\| = Mc_0\mu\|\xi_k - \xi_{k-1}\| \quad .$$

We now choose μ_0 so small that the inequality $Mc_0\mu_0 < \frac{1}{2}$ is satisfied. Then, taking account of (43), it follows that

$$\begin{aligned}\|\xi_{k+1} - \xi_k\| &\leq \frac{1}{2}\|\xi_k - \xi_{k-1}\| \leq \frac{1}{4}\|\xi_{k-1} - \xi_{k-2}\| \leq \cdots \\ &\leq \frac{1}{2^k}\|\xi_1 - \xi_0\| = \frac{1}{2^k}\|\xi_1\| \leq \frac{1}{2^{k+1}} c_1\mu^{n+1} \quad .\end{aligned} \tag{45}$$

Hence

$$\|\xi_{k+1}\| \le \|\xi_{k+1} - \xi_k\| + \|\xi_k - \xi_{k-1}\| + \dots + \|\xi_1\|$$
$$\le \left(\frac{1}{2^{k+1}} + \frac{1}{2^k} + \dots + \frac{1}{2}\right) c_1 \mu^{n+1} \le c_1 \mu^{n+1} \quad .$$

Thus, (44) and therefore, (45) are valid for any number k. By virtue of (45) the series of terms $\xi_{k+1} - \xi_k$ converges uniformly with respect to (t,s), that is, the sequence $\{\xi_k\}$ converges uniformly. This, in turn, implies the existence of a solution of the problem (34), (32). To see this, simply write equation (34) as an integral equation (from (37) it suffices to write y for ξ and α for G). The uniqueness of the solution of this integral equation can be proved in the usual way, if we note that the operator $B\xi + G(\xi,t,s,\mu)$ is Lipschitzian. Finally, the estimate (42) follows from the fact that each ξ_k satisfies the inequality (44). This concludes the proof of Theorem 4.

3. Concluding Remarks. 1. Our results can be extended, under certain additional assumptions, to the more general equation

$$\mu \frac{\partial x(t,s,\mu)}{\partial t} = -a(t,s)\left[x(t,s,\mu) - \int_a^b K(t,s,\sigma)x(t,\sigma,\mu)\,d\sigma\right] + \mu f\left(x,t,s,\int_a^b H(x,t,\sigma,\mu)\,d\sigma,\mu\right) \quad . \tag{46}$$

The most interesting feature of this equation is the appearance of the factor $a(t,s)$, whose dependence on s destroys the symmetry of the operator A .

Let us first investigate how to modify the construction in the special case when equation (46) is of the form

$$\mu\frac{\partial x(t,s,\mu)}{\partial t} = -a(s)[x(t,s,\mu) - \int_a^b K(s,\sigma)x(t,\sigma,\mu)d\sigma]$$
$$+ \mu f(x,t,s,\mu) \quad .$$

(This equation has been studied by A. Kasanov.) For $\bar{x}_0(t,s)$ and $\Pi_0 x(\tau,s)$ we have that

$$\bar{x}_0(t,s) = \sum_{i=1}^{k} \alpha_i(t)\varphi_i(s) \quad ,$$

$$\frac{\partial \Pi_0 x(\tau,s)}{\partial \tau} = -a(s)[\Pi_0 x(\tau,s) - \int_a^b K(s,\sigma)\Pi_0 x(\tau,\sigma)d\sigma] \quad ,$$

$$\Pi_0 x(0,s) = x^0(s) - \bar{x}_0(0,s) \quad ,$$

and

$$\Pi_0 x(\tau,s) \to 0 \quad \text{as} \quad \tau \to \infty \quad . \tag{47}$$

Condition (47) reduces to the condition that the initial value $\Pi_0 x(0,s)$ be bounded, that is,

$$\int_a^b \frac{\Pi_0 x(0,s)\varphi_i(s)}{a(s)} ds = 0 \quad (i = 1,\ldots,k)$$

(cf. (24)); whence, $\alpha_i(0)$ can be defined by the formula (cf. (25))

$$\alpha_i(0) = \int_a^b \frac{x^0(s)\varphi_i(s)}{a(s)} ds \quad (i = 1,\ldots,k) \quad .$$

We must of course choose the eigenfunctions $\varphi_i(s)$ $(i = 1,\ldots,k)$ to be orthogonal with weight $1/a(s)$.

The equations for the $\alpha_i(t)$ are obtained, as before, from a solvability condition in the equation for $\bar{x}_1(t,s)$, that is,

$$\frac{\partial\bar{x}_0(t,s)}{\partial t} = -a(s)[\bar{x}_1(t,s) - \int_a^b K(s,\sigma)\bar{x}_1(t,\sigma)d\sigma]$$

$$+ f(\bar{x}_0(t,s),t,s,0) ,$$

and they are

$$\frac{d\alpha_i}{dt} = \int_a^b \frac{f(\sum_{j=1}^k \alpha_j(t)\varphi_j(s),t,s,0)\varphi_i(s)}{a(s)} ds .$$

2. Equations and systems of the form $\mu\frac{dx}{dt} = A(t)x + \mu f$, where A is a certain integral operator, arise in kinetic theory (for example, Boltzmann's equation; see [6]). It is true that the structure of the operator A is more complicated in such problems than in the cases considered here; moreover, the problems are nonlinear. However, it is important to note the following: solutions of the degenerate equation $A(t)\bar{x} = 0$ contain an indeterminacy. In this regard we note that some of the approximation methods in kinetic theory (cf. [6]) lead to nonlinear systems of the type considered in Chapter 2.

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